Star points on smooth hypersurfaces

Filip Cools* and Marc Coppens**

March 11, 2009

Abstract. — *A point P on a smooth hypersurface X of degree d in $\mathbb{P}^N$ is called a star point if and only if the intersection of X with the embedded tangent space $T_P(X)$ is a cone with vertex P. This notion is a generalization of total inflection points on plane curves and Eckardt points on smooth cubic surfaces in $\mathbb{P}^3$. We generalize results on the configuration space of total inflection points on plane curves to star points. We give a detailed description of the configuration space for hypersurfaces with two or three star points. We investigate collinear star points and we prove that the number of star points on a smooth hypersurface is finite.*

MSC. — *14J70, 14N15, 14N20*

1 Introduction

In our papers [3] and [4], we studied the locus of smooth plane curves of degree d containing a given number of total inflection points. As a starting point in [3], we examined possible configurations of points and lines that can appear as such total inflection points and the associated tangent lines. In this paper, we generalize this point of view to higher dimensional varieties.

In case C is a smooth plane curve and P is a point on C, the intersection $T_P(C) \cap C$ of the tangent line with C is a 0-dimensional scheme of length d. The point P is called a total inflection point of C if and only if this scheme has maximal multiplicity (being d) at P. Therefore, as a generalization, we consider points P on smooth hypersurfaces X of degree d in $\mathbb{P}^N$ such that the intersection $T_P(X) \cap X$ of the tangent space with X has multiplicity d at P. This condition is equivalent to $T_P(X) \cap X$ being a cone with vertex P in $T_P(X)$. Because of this pictorial description, we call such point P a star point on X.

If P is a total inflection point of a plane curve C and $L = T_P(C)$, the intersection scheme $C \cap L$ is fixed: it is the divisor dP on L. Therefore, a

---

configuration space for total inflection points on plane curves is defined using pairs (L, P) with L a line on $\mathbb{P}^2$ and P a point on L. In case $N > 2$, P is a star point on a smooth hypersurface X and $\Pi = T_P(X)$, then $X \cap \Pi$ is not fixed in advance. Therefore, a configuration space for star points on hypersurfaces in $\mathbb{P}^N$ of degree d is defined using triples (Π, P, C) with Π a hyperplane in $\mathbb{P}^N$, P a point on Π and C a cone hypersurface of degree d in Π with vertex P. When restricting to smooth hypersurfaces (as done in this paper), one can assume C being smooth outside P.

As in the case of plane curves, we prove that there is a strong relation between the space of hypersurfaces having a given number e of star points and the space of associated configurations. Using this relation we find a lower bound for the dimension of the components of such space of hypersurfaces. We call this lower bound the expected dimension. We give a complete description of the configuration space associated to hypersurfaces X having two and three star points. In case $e = 2$, we have two components with one having the expected dimension and the other one having larger dimension. The component with unexpected dimension corresponds to the case when the line spanning two star points is contained in X. In case $e = 3$, we have components of the expected dimension but also components with larger dimension.

In case X is a smooth hypersurface of degree d in $\mathbb{P}^N$ and L is a line on X, then there are at most two star points of X on L. On the other hand, there do exist smooth hypersurfaces X in $\mathbb{P}^N$ of degree d such that for some line L the intersection $X \cap L$ consists of d star points of X. In case X is a smooth hypersurface of degree d in $\mathbb{P}^N$ and $L \not\subset X$ is a line containing at least $d-1$ star points of X, then L contains exactly d star points. This generalizes the classical similar fact on inflection points on cubic curves and it is an extra indication that the concept of star points is the correct generalization of total inflection points on plane curves. We show that the case with three star points and the case with collinear star points are the basic cases.

Although there exist smooth hypersurfaces having many star points (e.g. the Fermat hypersurfaces), we prove that there are only finitely many if $d \geq 3$.

Star points are intensively studied in the case of smooth cubic surfaces X in $\mathbb{P}^3$. In this case, a star point P is a point such that 3 lines on X meet at P. The study of the lines on a smooth cubic surface is very classical; it is well-known that there are exactly 27 lines on such a surface (see e.g. [8]). The study of smooth cubic surfaces having 3 lines through one point started with a paper by F.E. Eckardt (see [6]); therefore such points are often called Eckardt points. A classification of smooth cubic surfaces according to their star points (also considering the real case) was done in [14], using so-called harmonic homologies. In his Ph-D thesis [10], Nguyen gave a classification for the complex case using the description of a smooth cubic surface as a blowing-up of 6 points on $\mathbb{P}^2$ (see also [11]). From that point of view, the author also studied singular cubic surfaces (see also [12, 13]).

Eckardt points on cubic surfaces also appear in the recent paper [5]. For most of the smooth cubic surfaces S, there exists a $K3$-surface X being the minimal

smooth model of the quartic surface Y defined by the Hessian associated to the equation of S. In general, those $K3$-surfaces have Picard number 16 and one has an easy description for the generators of the Neron-Severi group $NS(X)$. Cubic surfaces S with Eckardt points give rise to higher Picard numbers. In particular, one also finds $K3$-surfaces with Picard number 20 (the so-called singular $K3$-surfaces). In those cases, the Eckardt points give rise to easily determined "new" curves on X.

The generalization to higher dimensional hypersurfaces also occurs in [2]. In this paper, the authors study the log canonical threshold of hyperplane sections for a smooth hypersurface X of degree N in $\mathbb{P}^N$. Note that such hypersurfaces are embedded by means of the anticanonical linear system. Assuming the log minimal model program, they obtain a strong relation between hyperplane sections coming from tangent spaces at star points of X and the minimal value for the log canonical threshold. Instead of talking about star points, the authors call them generalized Eckardt points.

A star point on a smooth hypersurface X in $\mathbb{P}^N$ of degree d gives rise to a subset of the Fano scheme $\mathbb{F}(X)$ of lines on X of dimension $N - 3$. The Fano scheme of hypersurfaces is intensively studied (see [1, 7, 9]). For a general hypersurface, the Fano scheme is smooth of dimension $2N - d - 3$. This also holds for non-general hypersurfaces if N is large with respect to d. On the other hand, in case $d > N$, the dimension of the Fano scheme of non-general hypersurfaces (e.g. Fermat hypersurfaces) might be larger than $2N - d - 3$. However, the finiteness of the number of star points for smooth hypersurfaces X in $\mathbb{P}^N$ of degree $d \geq 3$ implies that there exists no $(N-2)$-dimensional subset B of $\mathbb{F}(X)$ such that each line L of B meets a fixed curve $\Gamma \subset X$.

Star points are also related to so-called Galois points. Let X be a smooth hypersurface in $\mathbb{P}^N$, let P be a point in $\mathbb{P}^N$ and let H be a hyperplane not containing P. The point P is called a Galois point of X in case projection of X to H with center P corresponds to a Galois extension of function fields. In case P is an inner Galois point on X ("inner" means P is a point on X) then P is a star point of X. This is proved for quartic surfaces in [17, Corollary 2.2] and in general in [18, Corollary 6].

The article is structured as follows. In Section 2, we give the definition of a star point on a hypersurface and prove a general result on configurations of star points (Theorem 2.10). We use a connection between star points and polar hypersurfaces (Lemma 2.12) to determine all star points on a Fermat hypersurface (Example 2.13). In Section 3, we study star points on a line L in case $L \subset X$ (Proposition 3.1) and in case $L \not\subset X$ (Proposition 3.3 and Theorem 3.4). In Section 4, we prove that the number of star points on a smooth hypersurface is always finite (Theorem 4.2). In Section 5, we prove the lower bound for the dimension of the configuration space associated to a given number e of star points (Corollary 5.5). We also show that the case of collinear star points and three star points are basic (Proposition 5.7). In Section 6 and Section 7, we investigate the number of components of the configuration set of hypersurfaces with two or three star points.

2 Definition and first results

We work over the field $\mathbb{C}$ of complex numbers.

Notation 2.1. Let X be a hypersurface in $\mathbb{P}^N$ and let P be a smooth point on X. We write $T_P(X)$ to denote the tangent space of X inside $\mathbb{P}^N$.

Definition 2.2. Let X be a hypersurface of degree d in $\mathbb{P}^N$ ($N \geq 2$) and let P be a smooth point on X. We say that P is a *star point* on X if and only if the intersection $T_P(X) \cap X$ (as a scheme) has multiplicity d at P.

Remark 2.3. Some remarks related to the previous definition.

- Let X and P be as in Definition 2.2 and let $N \geq 3$. Then P is a star point of X if and only if the intersection $T_P(X) \cap X$ is a cone of degree d with vertex P inside $T_P(X)$. In particular, X has to be irreducible.

- Let C be a plane curve of degree d and let P be a smooth point on C. Then P is a star point of C if and only if P is a total inflection point for C (i.e. the tangent line to C at P intersects C with multiplicity d at P). Hence the concept of a star point is a generalization of the concept of a total inflection point of a plane curve.

- Let X be a surface of degree d in projective space $\mathbb{P}^3$ and let P be a star point on X. Then the intersection of the tangent plane to X at P with X is (as a set) a union of lines through P in that tangent plane. In case $d = 3$ and X is smooth, the intersection is the union of three lines on X through P (here we also use Lemma 2.4). Classically, such a point P is called an Eckardt point on X.

Lemma 2.4. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^N$ ($N \geq 3$) and let P be a star point on X. Then the intersection $X \cap T_P(X)$ is smooth outside the vertex P.*

Proof. Let $C = T_P(X) \cap X$ and assume Q is a singular point of C different from P. Since C is a cone with vertex P it follows that all points on the line $< P, Q >$ are singular points on C.

Choose coordinates $(X_0 : \ldots : X_N)$ in $\mathbb{P}^N$ such that $P = (1 : 0 : \ldots : 0)$, the hypersurface $T_P(X)$ has equation $X_N = 0$ and $Q = (1 : 1 : 0 : \ldots : 0)$. The equation of X is of the form

$$F(X_0, \ldots, X_N) = X_N G(X_0, \ldots, X_N) + H(X_1, \ldots, X_{N-1}),$$

with G (resp. H) homogeneous of degree $d-1$ (resp. d), $G(1, 0, \ldots, 0) \neq 0$ (because $Z(F)$ has to be smooth at P with tangent space $Z(X_N)$), $H(1, 0, \ldots, 0) = 0$ (because $Q \in T_P(X) \cap X$) and $(\partial H / \partial X_i)(1, 0, \ldots, 0) = 0$ for $1 \leq i \leq N-1$ (because C is singular at Q). Clearly $(a : b : 0 : \ldots : 0) \in X$ for all $(a : b) \in \mathbb{P}^1$. It follows that $(\partial F / \partial X_i)(1, c, 0, \ldots, 0) = 0$ for $0 \leq i \leq N-1$ and $(\partial F / \partial X_N)(1, c, 0, \ldots, 0) = G(1, c, 0, \ldots, 0)$. Unless $G(1, T, 0, \ldots, 0)$ is a

4

constant different from 0, it has a zero c_0 and then $(1 : c_0 : 0 : \ldots : 0)$ is a singular point on X, contradicting the assumption that X is smooth.

In case $G(1, T, 0, \ldots, 0)$ is a nonzero constant, one has

$$G = aX_0^{d-1} + \sum_{i=2}^{N} X_i G_i(X_0, \ldots, X_N),$$

with G_i homogeneous of degree $d - 2$ and $a \neq 0$. Consider the point $R = (0 : 1 : 0 : \ldots : 0) \in T_P(X) \cap X$. Still $(\partial F / \partial X_i)(R) = 0$ for $0 \leq i \leq N - 1$ but also $(\partial F / \partial X_N)(R) = G(R) = 0$, since $\partial F / \partial X_N = G + X_N \partial G / \partial X_N$. This implies X is singular at R, contradicting the assumptions. $\qquad\square$

In this paper we study star points on smooth hypersurfaces. Motivated by the previous lemma, we introduce the following definition.

Definition 2.5. Let $\mathbb{P}$ be some projective space and let P be a point in $\mathbb{P}$. A hypersurface C of degree d in $\mathbb{P}$ is called a *good P-cone* of degree d in $\mathbb{P}$ if C is a cone with vertex P and C is smooth outside P.

Using Definition 2.5, we can restate Lemma 2.4 as follows. If X is a smooth hypersurface of degree d in $\mathbb{P}^N$ and if P is a star point on X, then $T_P(X) \cap X$ as a scheme is a good P-cone of degree d in $T_P(X)$.

We are going to generalize Proposition 1.3 in [3] to the case of star points. First we introduce some terminology.

Notation 2.6. To a projective space $\mathbb{P}^N$, we associate the parameter space $\mathcal{P}_d$ of triples $\mathcal{T} = (\Pi, P, C)$ with Π a hyperplane in $\mathbb{P}^N$, P a point on Π and C a good P-cone of degree d in Π. Note that

$$\dim(\mathcal{P}_d) = N + (N - 1) + \binom{N + d - 2}{N - 2} - 1 = 2N + \binom{N + d - 2}{N - 2} - 2.$$

Consider such triples $\mathcal{T}_1, \ldots, \mathcal{T}_e$ with $\mathcal{T}_i = (\Pi_i, P_i, C_i)$ and $P_i \notin \Pi_j$ for $i \neq j$.

Definition 2.7. We say $\mathcal{L} = \{\mathcal{T}_1, \ldots, \mathcal{T}_e\}$ is *suited for degree d* if there exists a hypersurface X of degree d in $\mathbb{P}^N$ not containing Π_i for some $1 \leq i \leq e$ and such that $X \cap \Pi_i = C_i$.

Notation 2.8. Let $\mathcal{L} = \{\mathcal{T}_1, \ldots, \mathcal{T}_e\}$. We write

$$V_d(\mathcal{L}) = \{s \in \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \, | \, C_i \subset Z(s)\},$$

where $Z(s)$ is the zero locus of s. Let $\mathbb{P}_d(\mathcal{L}) = \mathbb{P}(V_d(\mathcal{L}))$ be its projectivization, so $\mathbb{P}_d(\mathcal{L})$ is a linear system of hypersurfaces of degree d in $\mathbb{P}^N$. For $\mathcal{L} = \emptyset$, we write $V_d = V_d(\mathcal{L})$ and $\mathbb{P}_d = \mathbb{P}_d(\mathcal{L})$ (so $V_d = \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$).

If Π is a hyperplane of $\mathbb{P}^N$, we can consider the restriction map

$$r : V_d(\mathcal{L}) \to \Gamma(\Pi, \mathcal{O}_\Pi(d)).$$

Denote the image of r by $V_{\Pi,d}(\mathcal{L})$ and write $\mathbb{P}_{\Pi,d}(\mathcal{L}) = \mathbb{P}(V_{\Pi,d}(\mathcal{L}))$ to denote its projectivization. Again, we can consider the case $\mathcal{L} = \emptyset$, and we write $V_{\Pi,d}(\mathcal{L}) = V_{\Pi,d}$ and $\mathbb{P}_{\Pi,d}(\mathcal{L}) = \mathbb{P}_{\Pi,d}$ in this case.

Note that $\dim(V_d) = \binom{N+d}{N}$ and $\dim(V_d(\Pi)) = \binom{N-1+d}{N-1}$.

Remark 2.9. In case X is a quadric in $\mathbb{P}^N$ and P is a smooth point on X, then $T_P(X) \cap X$ is a quadric in $T_P(X)$ singular at P. Since a singular quadric in some projective space is always a cone with vertex P, it follows that P is a star point of X. Therefore all smooth points on a quadric are star points on that quadric, hence from now on we only consider the case $d \geq 3$.

Theorem 2.10. *Assume $\mathcal{L}$ is suited using cones of degree $d \geq 3$.*

(i) *Then $\dim(\mathbb{P}_d(\mathcal{L})) = \binom{d-e+N}{N}$ (here $\binom{d-e+N}{N} = 0$ if $e > d$). In case $e \leq d$, a general element X of $\mathbb{P}_d(\mathcal{L})$ is a smooth hypersurface of degree d.*

(ii) *Let Π be a hyperplane in $\mathbb{P}^N$ with $P_i \notin \Pi$ for $1 \leq i \leq e$. Let $\Pi'_i = \Pi \cap \Pi_i$ for $1 \leq i \leq e$.*

If $e \leq d$, then $\mathbb{P}_{\Pi,d}(\mathcal{L})$ has dimension $\binom{d-e+N-1}{N-1}$ and it contains

$$\Pi'_1 + \ldots + \Pi'_e + \mathbb{P}_{\Pi,d-e}.$$

If $e > d$, then $\mathbb{P}_{\Pi,d}(\mathcal{L})$ has dimension 0.

(iii) *Let Π be a hyperplane as before. Fix a point P in Π and a good P-cone C of degree d in Π. Let $\mathcal{T} = (\Pi, P, C)$ and consider $\mathcal{L}' = \mathcal{L} \cap \{\mathcal{T}\}$. Then $\mathcal{L}'$ is suited for degree d if and only if $C \in \mathbb{P}_{\Pi,d}(\mathcal{L})$.*

Proof. First we are going to prove that (ii) and (iii) follow from (i). So assume $\mathcal{L}$ is suited and (i) holds for $\mathcal{L}$. Let Π be a hyperplane in $\mathbb{P}^N$ such that $P_i \notin \Pi$ for $1 \leq i \leq e$.

First assume $\dim(\mathbb{P}_d(\mathcal{L})) = 0$, i.e. $\mathbb{P}_d(\mathcal{L})$ contains a unique hypersurface X of degree d not containing Π_i for $1 \leq i \leq e$. It follows that $e > d$ and $\Pi_i \cap X = C_i$ for $1 \leq i \leq e$. If $\Pi \subset X$ then Π'_i is a hyperplane inside Π_i not containing P_i and contained in X. This contradicts the fact that $C_i = X \cap \Pi_i$ because C_i is a cone in Π_i with vertex P_i. It follows that $\Pi \not\subset X$, hence $\mathbb{P}_{\Pi,d}(\mathcal{L})$ is not empty. Clearly it has dimension zero, thus (ii) holds. Let P, C, $\mathcal{T}$ $\mathcal{T}'$ be as described in (iii). If $X' \in \mathbb{P}_d(\mathcal{L}')$, then clearly $X' \in \mathbb{P}_d(\mathcal{L})$, hence $X' = X$ and we need $X \cap \Pi = C$ (we already know that $\Pi \subset X$ is impossible), so $C \in \mathbb{P}_{\Pi,d}(\mathcal{L})$. Conversely, if $C \in \mathbb{P}_{\Pi,d}(\mathcal{L})$, then we have $X \cap \Pi = C$, hence $X \in \mathbb{P}_d(\mathcal{L}')$. Since we already proved that $\Pi \not\subset X$ it follows that $\mathcal{L}'$ is suited. This proves (iii) in this case.

Now assume $e \leq d$, hence $\dim(\mathbb{P}_d(\mathcal{L})) = \binom{d-e+N}{N}$. Let Π be as in (ii) and let $s \in V_d(\mathcal{L})$ with $r(s) = 0$ (where r is as in Notation 2.8), hence $\Pi \subset Z(s)$. It follows that $\Pi'_i \subset Z(s)$ but also $C_i \subset \Pi_i \cap Z(s)$. Since P_i is a vertex of the cone C_i and $P_i \notin \Pi'_i$, it follows that $\Pi_i \subset Z(s)$. This proves that

$$\ker(r) = \pi_1 \cdots \pi_e . \pi . \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-e-1)),$$

where we write π to denote an equation of Π and so on, hence $\dim(\ker(r)) = \binom{d-e+N-1}{N}$. This implies

$$
\begin{aligned}
\dim(\mathrm{im}(r)) &= \dim(V_d(\mathcal{L})) - \dim(\ker(r)) \\
&= \binom{d-e+N}{N} + 1 - \binom{d-e+N-1}{N} = \binom{d-e+N-1}{N-1} + 1,
\end{aligned}
$$

hence $\dim(\mathbb{P}_{\Pi,d}(\mathcal{L})) = \binom{d-e+N-1}{N-1}$. Since

$$
\pi_1.\cdots.\pi_e.\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-e)) \subset V_d(\mathcal{L}),
$$

we have $\Pi'_1 + \ldots + \Pi'_e + \mathbb{P}_{\Pi,d-e} \subset \mathbb{P}_{\Pi,d}(\mathcal{L})$. This finishes the proof of (ii).

Let $P \in \Pi$ with $P_i \notin \Pi$ and C a good P-cone of degree d in Π. Let $\mathcal{T} = (\Pi, P, C)$ and assume $\mathcal{L}' = \mathcal{L} \cup \{\mathcal{T}\}$ is suited. There exists $X' \in \mathbb{P}_d(\mathcal{L}')$ with $X' \cap \Pi = C$. Since $X' \in \mathbb{P}_d(\mathcal{L})$, it follows $C \in \mathbb{P}_{\Pi,d}(\mathcal{L})$. Conversely, assume $C \in \mathbb{P}_{\Pi,d}(\mathcal{L})$, hence there exists $X \in \mathbb{P}_d(\mathcal{L})$ with $X \cap \Pi = C$. Assume X contains a hyperplane Π_i for some $1 \le i \le e$, then $\Pi \cap \Pi_i \subset X$. Since $P \notin \Pi_i$ and $X \cap \Pi$ is a cone with vertex P different from Π, we obtain a contradiction. Therefore $\Pi_i \not\subset X$. This proves $\mathcal{L}'$ is suited, finishing the proof of (iii).

Now we are going to prove (i). First we consider the case $e = 1$. Consider the exact sequence

$$
0 \to \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-1)) \to \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to \Gamma(\Pi_1, \mathcal{O}_{\Pi_1}(d)) \to 0.
$$

The cone $C_1 \subset \Pi_1$ corresponds to a subspace of $V_{\Pi_1,d} = \Gamma(\Pi_1, \mathcal{O}_{\Pi_1}(d))$ of dimension one and its inverse image in $V_d = \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ is equal to $V_d(\{\mathcal{T}_1\})$. This proves $\dim(\mathbb{P}_d(\{\mathcal{T}_1\}) = \binom{N+d-1}{d}$ and for $X \in \mathbb{P}_d(\{\mathcal{T}_1\})$ general, one has $X \cap \Pi_1 = C_1$. This finishes the proof of (i) in case $e = 1$. Moreover, we conclude that each $\{\mathcal{T}_1\}$ is suited.

Now assume $e > 1$ and assume $\mathcal{L}$ is suited and let $\mathcal{L}' = \{\mathcal{T}_1, \ldots, \mathcal{T}_{e-1}\}$. Since $\mathcal{L}'$ is suited we can assume (i) holds for $\mathcal{L}'$ (using the induction hypothesis). First assume $e - 1 > d$, hence $\dim(\mathbb{P}_d(\mathcal{L}')) = 0$. Since $\mathbb{P}_d(\mathcal{L}) \neq \emptyset$ and $\mathbb{P}_d(\mathcal{L}) \subset \mathbb{P}_d(\mathcal{L}')$, it follows $\dim(\mathbb{P}_d(\mathcal{L})) = 0$, hence (i) holds for $\mathcal{L}$. So assume $e - 1 \le d$, hence

$$
\dim(\mathbb{P}_d(\mathcal{L}')) = \binom{d-e+1+N}{N}.
$$

In particular (iii) holds for $\mathcal{L}'$ and we apply it to $\Pi = \Pi_e$. It follows that $C_e \in \mathbb{P}_{\Pi_e,d}(\mathcal{L}')$, hence C_e defines a one-dimensional subspace of $V_{\Pi_e,d}(\mathcal{L}')$. The inverse image under r is exactly $V_d(\mathcal{L}) \subset V_d(\mathcal{L}')$. Since

$$
\pi_1.\cdots.\pi_e.\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-e)) = \ker(r),
$$

it follows that $\ker(r) = 0$ if $e = d + 1$. This implies $\dim V_d(\mathcal{L}) = 1$ in that case, hence $\dim(\mathbb{P}_d(\mathcal{L})) = 0$. In case $e \le d$, we find $\dim(\ker(r)) = \binom{d-e+N}{N}$ and so $\dim(V_d(\mathcal{L})) = \binom{d-e+N}{N} + 1$.

Finally, in case $e \le d$, we are going to prove that a general $X \in \mathbb{P}_d(\mathcal{L})$ is smooth. We know that $\Pi_1 + \ldots + \Pi_e + \mathbb{P}_{d-e} \subset \mathbb{P}_d(\mathcal{L})$. For

$$
X \in \mathbb{P}_d(\mathcal{L}) \setminus (\Pi_1 + \ldots + \Pi_e + \mathbb{P}_{d-e}),
$$

7

one has $X \cap \Pi_i = C_i$ for $1 \leq i \leq e$. (Indeed, assume there exists $1 \leq i \leq e$ such that $\Pi_i \subset X$. Then for $j \neq i$, consider $\Pi_i \cap \Pi_j \subset X$. But $C_j \subset \Pi_j \cap X$ and C_j is a good P-cone of degree d and $P_j \notin \Pi_i$, so it follows $\Pi_j \subset X$, hence $X \in \Pi_1 + \ldots + \Pi_e + \mathbb{P}_{d-e}$.) Hence the fixed locus of $\mathbb{P}_d(\mathcal{L})$ is equal to $C_1 \cup \ldots \cup C_e$. From Bertini's Theorem it follows that $\mathrm{Sing}(X) \subset C_1 \cup \ldots \cup C_e$ for general $X \in \mathbb{P}_d(\mathcal{L})$. Since for $X \notin \Pi_1 + \ldots + \Pi_e + \mathbb{P}_{d-e}$, one has $\Pi_i \cap X = C_i$, so it follows that $\mathrm{Sing}(X) \cap C_i \subset \{P_i\}$. Therefore, if a general $X \in \mathbb{P}_d(\mathcal{L})$ would be singular, then there exists some $1 \leq i \leq e$ such that X is singular at P_i. This implies P_i is a singular point of all $X \in \mathbb{P}_d(\mathcal{L})$. However, inside $\Pi_1 + \ldots + \Pi_e + \mathbb{P}_{d-e}$, there exists elements of $\mathbb{P}_d(\mathcal{L})$ smooth at P_i for all $1 \leq i \leq e$ (here we used $e \leq d$). Hence a general $X \in \mathbb{P}_d(\mathcal{L})$ is smooth. $\qquad\square$

To finish this section, we mention an indirect characterization of star points that is already mentioned in the old paper [15]. We start by recalling the following classical definition.

Definition 2.11. Let X be a hypersurface of degree d in $\mathbb{P}^r$ with equation $F = 0$ and let $P = (x_0 : \ldots : x_r)$ be a point in $\mathbb{P}^r$. The *polar hypersurface* of P with respect to X is the hypersurface $\Delta_P(X)$ defined by the equation $\sum_{i=0}^r x_i(\partial F/\partial X_i) = 0$. Clearly $Q \in X \cap \Delta_P(X)$ with $Q \neq P$ if and only if $\langle P, Q \rangle \subset T_Q(X)$.

The following easy lemma shows how polar hypersurfaces can be used to find star points. Although the proof can be given using equations, we prefer to give geometric arguments.

Lemma 2.12. *Let X be a smooth hypersurface in $\mathbb{P}^N$ and let $P \in \mathbb{P}^N$, then $\Delta_P(X)$ contains a hyperplane Π of $\mathbb{P}^N$ containing P if and only if P is a star point of X and $\Pi = T_P(X)$.*

Proof. In case P is a star point on X and $Q \in T_P(X) \cap X$ different from P, then $\langle P, Q \rangle \subset X$. Hence $\langle P, Q \rangle \subset T_Q(X)$ and so $Q \in \Delta_P(X)$. This proves

$$T_P(X) \cap X \subset \Delta_P(X) \cap T_P(X).$$

If $T_P(X) \not\subset \Delta_P(X)$, we have $\deg(T_P(X) \cap X) = d$ and $\deg(\Delta_P(X) \cap T_P(X)) = d - 1$, a contradiction. We conclude $T_P(X) \subset \Delta_P(X)$.

Conversely, assume there exists a hyperplane Π in $\mathbb{P}^r$ containing P and satisfying $\Pi \subset \Delta_P(X)$. If Q is a singular point of $\Pi \cap X$, then $\Pi = T_Q(X)$. Since X is smooth, the equality holds at most for finitely many points (see e.g. [19, Corollary 2.8]). Therefore $\Pi \cap X$ has no multiple components. For each component Y of $\Pi \cap X$ and $Q \in Y$ general, one finds $\langle P, Q \rangle$ is tangent to Y at Q. Hence the projection with center P in Π restricted to Y does not have an injective tangent map. Because of Sard's Lemma, this is only possible in case this projection has fibers of dimension at least one. So we conclude that Y is a cone with vertex P. It follows that P is a star point on X. $\qquad\square$

Example 2.13. Consider the Fermat hypersurface $X_{d,N} = Z(X_0^d + \ldots + X_N^d) = Z(F_{d,N}) \subset \mathbb{P}^N$. Let $\xi \in \mathbb{C}$ with $\xi^d = -1$ and let $E_{i,j}(\xi)$ be the point with $x_i = 1$,

$x_j = \xi$ and $x_k = 0$ for $k \neq i, j$. Clearly, $E_{i,j}(\xi) \in X_{d,N}$. We are going to show that $E_{i,j}(\xi)$ is a star point on $X_{d,N}$.

From $\partial F_{d,N}/\partial X_k = d.X_k^{d-1}$, we find $(\partial F_{d,N}/\partial X_k)(E_{i,j}(\xi)) = 0$ in case $k \notin \{i,j\}$, $(\partial F_{d,N}/\partial X_i)(E_{i,j}(\xi)) = d$ and $(\partial F_{d,N}/\partial X_j)(E_{i,j}(\xi)) = d.\xi^{d-1} = -d.\xi^{-1}$, hence $T_{E_{i,j}(\xi)}(X)$ has equation $d(X_i - 1) - d.\xi^{-1}(X_j - \xi) = 0$, hence $X_i - \xi^{-1} X_j = 0$. In order to compute $T_{E_{i,j}(\xi)}(X) \cap X$, we replace X_i by $\xi^{-1} X_k$ in the equation of $X_{d,N}$ and one finds $\sum_{k=0; k \notin \{i,j\}}^{N} X_k^d = 0$. Inside $T_{E_{i,j}(\xi)}(X)$, this is a cone with vertex $E_{i,j}(\xi)$. So we find smooth hypersurfaces in $\mathbb{P}^N$ containing a lot of star points.

Using Lemma 2.12, we can prove that we found all star points on $X_{d,N}$. Indeed for $P = (x_0 : \ldots : x_N)$, one has that the polar hypersurface $\Delta_P(X_{d,N})$ has equation $\sum_{i=0}^{N} x_i X_i^{d-1} = 0$. Clearly,

$$\mathrm{Sing}(\Delta_P(X_{d,N})) = Z(X_k \,|\, x_k \neq 0).$$

If P is a star point then $\Delta_P(X_{d,N})$ should contain a hyperplane, therefore the singular locus of $\Delta_P(X_{d,N})$ has dimension $N - 2$. This implies that there exist $0 \leq i < j \leq N$ such that $x_k = 0$ for $k \notin \{i,j\}$. Since there is no point on $X_{d,N}$ having N coordinates equal to 0 we can assume $x_i = 1$. Since $P \in X$ we need $x_j^d = -1$. So, the hypersurface $X_{d,N}$ has exactly $d\binom{N+1}{2}$ star points.

Remark 2.14. On a smooth cubic surface X in $\mathbb{P}^3$, there are exactly 27 lines. Each of these lines contains at most two star points (see Proposition 3.1) and each star point gives rise to three lines, hence the number of star points on such a surface X is at most 18. This upper bound is attained by the Fermat surface $X_{3,3} \subset \mathbb{P}^3$.

3 Collinear star points

Let X be a smooth hypersurface in $\mathbb{P}^N$ and let P_1, P_2 be two different star points on X. Assume $P_2 \in T_{P_1}(X)$. Since $T_{P_1}(X) \cap X$ is a cone with vertex P_1, it follows that the line $L = \langle P_1, P_2 \rangle$ is contained in X. We investigate star points on X belonging to a line $L \subset X$.

Proposition 3.1. *Let X be a smooth hypersurface of degree $d \geq 3$ in $\mathbb{P}^N$ and let L be a line in $\mathbb{P}^N$ such that $L \subset X$. Then X has at most two star points on the line L.*

Proof. From the assumptions, it follows that $N \geq 3$.

We first consider the case $N = 3$, thus L is a divisor on X. For a hyperplane Π in $\mathbb{P}^3$, we write $\Pi \cap X$ to denote the effective divisor on X corresponding to Π (hence it is the intersection considered as a scheme). We consider the pencil $\mathbb{P} = \{\Pi \cap X - L \,|\, \Pi \text{ is a plane in } \mathbb{P}^3 \text{ containing } L\}$ on X.

Assume there exist $D \in \mathbb{P}$ with $L \subset D$, hence there is a plane $\Pi \subset \mathbb{P}^3$ with $\Pi \cap X \geq 2L$. It follows that $\Pi = T_P(X)$ for all $P \in L$. If X would have a star point P on L then any $Q \in L \setminus \{P\}$ would be a singular point of $T_P(X) \cap X$, this

is impossible because of Lemma 2.4. So, since for each $D \in \mathbb{P}$ one has $L \not\subset D$, the pencil $\mathbb{P}$ induces a g_{d-1}^1 on L.

If $P \in L$ is a star point of X, then there is a divisor $L_1 + \ldots + L_{d-1}$ in $\mathbb{P}$ with $L_1, \ldots, L_{d-1}$ lines through P different from L, hence $(d-1)P \in g_{d-1}^1$. Since g_{d-1}^1 on $\mathbb{P}^1$ can have at most two total ramification points, it follows that X has at most two star points on L. (Note that here we use $d \geq 3$. Indeed, this lemma clearly does not hold for quadrics in $\mathbb{P}^3$).

Now assume $N > 3$ and assume the proposition holds in $\mathbb{P}^{N-1}$. Consider the linear system $\mathbb{P} = \{X \cap \Pi \,|\, \Pi$ is a hyperplane in $\mathbb{P}^N$ containing $L\}$ in $\mathbb{P}^N$. It has dimension $N-2$ and its fixed locus is L. It follows from Bertini's Theorem that for a general $D \in \mathbb{P}$ one has $\mathrm{Sing}(D) \subset L$. However, in case $P \in L$ is a singular point of $D \in \mathbb{P}$, then $D = X \cap T_P(X)$, hence the locus of divisors $D \in \mathbb{P}$ singular at some point of L has at most dimension one. Since $N-2 > 1$, it follows that a general $D \in \mathbb{P}$ is smooth.

So take a general hyperplane Π in $\mathbb{P}^N$ containing L and consider

$$X' = X \cap \Pi \subset \Pi \cong \mathbb{P}^{N-1},$$

again a smooth hypersurface. If P is a star point for X on L, then by definition it is also a star point for X'. By induction, we know X' has at most two star points on L, hence X has at most two star points on L. $\qquad\square$

Example 3.2. The affine surface X in $\mathbb{A}^3$ with equation

$$yx(y-x)(y+x) + z(x-1)(z+x-1)(z-x+1) = 0$$

has degree equal to 4 and contains the x-axis. On this axis, there are two star points, namely $P_1 = (0,0,0)$ with $T_{P_1}X : z = 0$ and $P_2 = (1,0,0)$ with $T_{P_2}X : y = 0$.
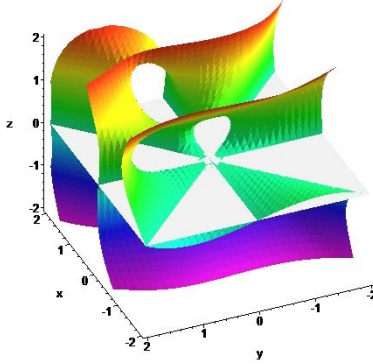

Figure 1: Star point P_1

On Figure 1, one can see that the lines $L_1 : y = 0$ (i.e. the x-axis), $L_2 : x = 0$, $L_3 : y = x$ and $L_4 : y = -x$ on $T_{P_1}X$ are contained in X.

Now we consider star points on a line L not contained in the smooth hypersurface X. Of course, such a line L contains at most d star points of X. The following result shows this upper bound occurs.

Proposition 3.3. *Fix integers $d \geq 3$ and $N \geq 3$. Let L be a line in $\mathbb{P}^N$ and let $P_1, \ldots, P_d$ be different points on L. Fix hyperplanes $\Pi_1, \ldots, \Pi_d$ not containing L and such that $P_i \in \Pi_i$ for $1 \leq i \leq d$ and fix a good P_1-cone C_1 in Π_1 of degree d. Then there exists a smooth hypersurface X of degree d in $\mathbb{P}^N$ not containing L, having a star point at $P_1, \ldots, P_d$ with $T_{P_i}(X) = \Pi_i$ for $1 \leq i \leq d$ and $T_{P_1}(X) \cap X = C_1$.*

Proof. Let Y be a cone on C_1 with vertex in L different from $P_1, \ldots, P_d$. For $2 \leq i \leq d$, let $Y \cap \Pi_i = C_i$. Clearly, C_i is a good P_i-cone in Π_i. Let $\mathcal{T}_i = (\Pi_i, P_i, C_i)$ for $1 \leq i \leq d$ and denote $\mathcal{L} = \{\mathcal{T}_1, \ldots, \mathcal{T}_d\}$. Since $Y \in \mathbb{P}_d(\mathcal{L})$ and $\Pi_i \not\subset Y$ for $1 \leq i \leq d$, we find $\mathcal{L}$ is suited for degree d. From Theorem 2.10, it follows a general element X of $\mathbb{P}_d(\mathcal{L})$ is smooth. $\square$

For smooth cubic plane curves C, there is the following classical result: if P_1, P_2 are inflection points of C, then the third intersection point of C with the line connecting P_1 and P_2 is also an inflection point. For plane curves, this result has a generalization. Indeed, let C be a smooth plane curve of degree $d \geq 2$, let L be a line and assume $P_1, \ldots, P_{d-1}$ are total inflection points of C contained in L. Then C has one more intersection point P_d with L and P_d is also a total inflection point of C. We generalize this to the case of star points. This result is an extra indication that the concept of star point is the correct generalization of the concept of total inflection point.

Theorem 3.4. *Let $N \geq 3$ be an integer and let X be a smooth hypersurface of degree $d \geq 3$ in $\mathbb{P}^N$. Let L be a line in $\mathbb{P}^N$ with $L \not\subset X$ and assume $P_1, \ldots, P_{d-1}$ are $d-1$ different star points of X on L. Then L intersects X transversally at d points $P_1, \ldots, P_d$ and P_d is also a star point of X.*

Proof. Since $T_{P_i}(X) \cap X$ is a good P_i-cone C_i and $L \not\subset X$, it follows that $L \not\subset T_{P_i}(X)$. In particular, L intersects X transversally at P_i for $1 \leq i \leq d-1$, hence L intersects X at d different points $P_1, \ldots, P_d$. Let $\Pi_i = T_{P_i}(X)$ for $1 \leq i \leq d$.

We need to prove that P_d is a star point of X. Since $L \not\subset \Pi_d$ it follows that $\Pi_d \cap \Pi_i$ is a hyperplane in Π_i not containing the vertex of the P_i-good cone C_i. This implies $C_{i,d} = C_i \cap \Pi_d$ is a smooth hypersurface of degree d in $\Pi_i \cap \Pi_d$. Take $i = 1$ and let $P \in C_{1,d}$ and consider the plane $\Lambda_P = \langle P, L \rangle$ spanned by P and L. Since $L \not\subset \Pi_d$ and $P_d, P \in \Pi_d$, it follows that the line $\langle P_d, P \rangle$ is equal to $\Lambda_P \cap \Pi_d$. Since $P \in X \cap \Pi_1$, it follows that the line $\langle P, P_1 \rangle$ belongs to X, hence $\langle P, P_1 \rangle \subset \Lambda_P \cap X$. Let $Q_i = \langle P_1, P \rangle \cap \Pi_i$ for $2 \leq i \leq d-1$, then $Q_i \in X \cap \Pi_i$ while $P_i \neq Q_i$, hence $\langle P_i, Q_i \rangle \subset C_i = X \cap \Pi_i$ and $\langle P_i, Q_i \rangle \subset \Lambda_P \cap X$. So we obtain $d-1$ different lines on $X \cap \Lambda_P$, hence

$$\langle P_1, P \rangle + \langle P_2, Q_2 \rangle + \ldots + \langle P_{d-1}, Q_{d-1} \rangle \subset \Lambda_P \cap X$$

Since $\Lambda_P \not\subset X$, we know that $\Lambda_P \cap X$ is a curve (effective divisor) of degree d on Λ_P. Hence $\Lambda_P \cap X$ needs to be a sum of d lines in Λ_P. Since

$$P_d \notin \langle P_1, P \rangle \cup \langle P_2, Q_2 \rangle \cup \ldots \cup \langle P_{d-1}, Q_{d-1} \rangle,$$

it follows that $\Lambda_P \cap X$ contains a line T through P_d. Since $T \subset X$, it follows that $T \subset \Pi_d$, hence $T \subset \Pi_d \cap \Lambda_P$ and therefore $T = \langle P_d, P \rangle$. Since P is any point on $C_{1,d}$ it follows that the cone C_d on $C_{1,d}$ with vertex P_d is contained in $\Pi_d \cap X$. However this cone is a hypersurface of degree d in $\Pi_d \cap X$, hence $C_d = \Pi_d \cap X$. Since C_d is a good P_d-cone in Π_d, it follows that P_d is a star point on X. $\qquad\square$

4 The number of star points

The number of total inflection points on a smooth plane curve is bounded because a total inflection point gives some contribution to the divisor corresponding to the inflection points of the associated two-dimensional linear system. The degree of this divisor is fixed by the degree of the plane curve. For the case of star points on hypersurfaces of dimension at least two, such argument is not available.

Although there exist hypersurfaces having a large number of star points (see Proposition 3.3 or Example 2.13), we prove the finiteness of the number of star points on a smooth hypersurface. First we prove that the locus of star points form a Zariski-closed subset. This implies that a smooth hypersurfaces X having infinitely many star points should contain a curve Γ such that each point on that curve is a star point of X.

Lemma 4.1. *Let $X \subset \mathbb{P}^N$ be a smooth hypersurface of degree $d \geq 3$ and let $ST(X)$ be the set of star points on X. Then the set $ST(X)$ is a Zariski-closed subset of X.*

Proof. Let $(\mathbb{P}^N)^*$ be the dual space of $\mathbb{P}^N$ (parameterizing hyperplanes in $\mathbb{P}^N$) and let $\mathcal{H} \subset (\mathbb{P}^N)^* \times \mathbb{P}^N$ be the incidence space (as a set it is defined by $(\Pi, P) \in \mathcal{H}$ if and only if $P \in \Pi$; clearly it is Zariski-closed in $(\mathbb{P}^N)^* \times \mathbb{P}^N$). Consider also the Zariski-closed subset $\widetilde{X} \subset X \times (\mathbb{P}^N)^*$ (isomorphic to X) with $(P, \Pi) \in \widetilde{X}$ if and only if $\Pi = T_P(X)$. Then $\mathbb{P}(T_X)$ can be identified with

$$(\widetilde{X} \times \mathbb{P}^N) \cap (X \times \mathcal{H}) \subset X \times (\mathbb{P}^N)^* \times \mathbb{P}^N.$$

The projection $p : \mathbb{P}(T_X) \to X$ is a $\mathbb{P}^{N-1}$-bundle and $\mathcal{X} = (\widetilde{X} \times X) \cap (X \times \mathcal{H})$ is a divisor in $\mathbb{P}(T_X)$ giving a family $p' : \mathcal{X} \to X$ of hypersurfaces of degree d for p. There is a natural section $s : X \to \mathcal{X}$ with $s(P) = (P, T_P(X), P)$. Using local equations, it is clear that the set of points Q on $\mathcal{X}$ such that $p'^{-1}(p'(Q))$ has multiplicity d at Q is a Zariski-closed subset Y of $\mathcal{X}$. Since $p'(Y \cap s(X)) = ST(X)$ and p' is proper, it follows that $ST(X)$ is Zariski-closed in X. $\qquad\square$

The proof of the following theorem is inspired by [9]. In our situation, it is possible to work a bit more geometrically.

Theorem 4.2. *Let X be a smooth hypersurface of degree $d \geq 3$ in $\mathbb{P}^n$, then X has at most finitely many star points.*

Proof. Assume X has infinitely many star points, then there exists a curve Γ on X such that each $P \in \Gamma$ is a star point on X, so $C_P = X \cap T_P(X)$ is a cone with vertex P.

Let $\mathbb{G} = \mathbb{G}(1, N)$ be the Grassmannian of lines in $\mathbb{P}^N$. For a line $L \subset \mathbb{P}^N$, we will denote the corresponding point in $\mathbb{G}$ by l. For each star point $P \in \Gamma$, the set of lines in X through P gives rise to a subset $\mathcal{C}_P \subset \mathbb{G}$ of dimension $N - 3$. By moving P on Γ and $\mathcal{C}_P$ inside $\mathbb{G}$, we obtain a set $B \subset \mathbb{G}$ with $\dim(B) = N - 2$.

Let P be a general point on Γ, pick l general in $\mathcal{C}_P$ and let Q be a general point on L. Choose coordinates $(X_0 : \ldots : X_N)$ on $\mathbb{P}^N$ such that $P = E_0$, $Q = E_1$ and $T_P(X) : X_2 = 0$, where E_i is the point with zero coordinates except the ith coordinate being one. Write $T = \cap_{R \in L} T_R(X)$. Note that T is a linear space of dimension equal to $N - 2$ or $N - 1$, since $T_Q(C_P) \subset T \subset T_Q(X)$.

Assume $\dim(T) = N - 1$. In this case, for each $R \in L$, the tangent space $T_R(X)$ has equation $X_2 = 0$, thus $\frac{\partial F}{\partial X_i}(R) = 0$ for all $i \neq 2$. On the other hand, there is at least one point $R^\star \in L$ such that $\frac{\partial F}{\partial X_2}(R^\star) = 0$. This implies $R^\star$ is a singular point of X, a contradiction.

Now assume $\dim(T) = N - 2$. Note that $T_P(\Gamma) \not\subset T$. Indeed, otherwise we have $T_P(\Gamma) \subset T_S(C_P)$ for all $S \in C_P$, hence, from Sard's Lemma, it follows that the projection of C_P from $T_P(\Gamma)$ has 2-dimensional fibers. Because it is a projection, those fibers are planes containing $T_P(\Gamma)$ and C_P is a cone with vertex $T_P(\Gamma)$. Since P is the only singular point of C_P, this is impossible.

So we can choose the coordinates on $\mathbb{P}^N$ so that $T_P(\Gamma) = \langle P, E_3 \rangle$ and $T_Q(X)$ has equation $X_3 = 0$, thus T has equation $X_2 = X_3 = 0$. For each $R \in L$, we have $\frac{\partial F}{\partial X_i}(R) = 0$ for all $i \notin \{2, 3\}$ and the tangent space $T_R(X)$ has equation $\frac{\partial F}{\partial X_2}(R)X_2 + \frac{\partial F}{\partial X_3}(R)X_3 = 0$.

Let $\mathbb{P}^N = \mathbb{P}(W)$ for some $(N + 1)$-dimensional vector space W and let $\{e_0, \ldots, e_N\}$ be a basis of W such that $E_i = [e_i]$. Since $T_Q(X)$ has equation $X_3 = 0$, there exists a holomorphic arc $\{q(t)\} \subset W$ with $Q(t) = [q(t)] \in X$, $q(0) = e_1$ (so $Q = [q(0)]$) and $q'(0) = e_2$. Note that the arc $\{Q(t)\} \subset X$ is not contained in $T_P(X)$. Since the lines in B cover X, there exists an arc $\{p(t)\} \subset W$ such that $P(t) = [p(t)] \in \Gamma$, $P(0) = P$ and the lines $\langle P(t), Q(t) \rangle$ are contained in B. Write $p'(0) = \lambda.e_3$ with $\lambda \in \mathbb{C}$.

Now let $R = (a : b : 0 : \ldots : 0)$ be general point on L, so $\{r(t)\} \subset W$ with $r(t) = a.p(t) + b.q(t)$ is a holomorphic arc with $R(t) = [r(t)] \in X$ and $R(0) = R$. We have $r'(0) = a\lambda.e_3 + b.e_2$, hence

$$[r(0) + r'(0)] = (a : b : b : a\lambda : 0 : \ldots : 0) \in T_R(X).$$

If $\lambda = 0$, we have $\frac{\partial F}{\partial X_2}(R) = 0$ for all point $R \in L \setminus \{P\}$, and thus $\frac{\partial F}{\partial X_2}(P) = 0$, a contradiction. So $\lambda \neq 0$ and we conclude $\frac{\partial F}{\partial X_2}X_1 + \lambda\frac{\partial F}{\partial X_3}X_0 = 0$ on L. This implies the existence of a homogeneous form $\check{G}(X_0, X_1)$ of degree $d - 2$ such that $\frac{\partial F}{\partial X_2} = \lambda X_0 G$ and $\frac{\partial F}{\partial X_3} = -X_1 G$ on L. Choose $(a^\star, b^\star) \neq (0, 0)$ such

that $G(a^\star, b^\star) = 0$. This corresponds to a point $R^\star \in L$ satisfying $\frac{\partial F}{\partial X_2}(R^\star) = \frac{\partial F}{\partial X_3}(R^\star) = 0$. Hence $R^\star$ is a singular point on X, again a contradiction. $\qquad\square$

Corollary 4.3. *Let X be a smooth hypersurface in $\mathbb{P}^N$ of degree $d \geq 3$ and let $\mathbb{F}(X)$ be its Fano scheme of lines. Then there does not exist a $(N-2)$-dimensional subset $B \subset \mathbb{F}(X)$ and a curve $\Gamma \subset X$ such that each line of B meets Γ.*

Proof. Since a smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 3$ contains only finitely many lines, the statement is clear for $N = 3$. Now assume the statement fails for some $N > 3$. So, there exists a smooth hypersurface $X \subset \mathbb{P}^N$ of degree $d \geq 3$, a subset $B \subset \mathbb{F}(X)$ of dimension $N - 2$ and a curve $\Gamma \subset X$ such that each line of B meets Γ. We will denote the line corresponding to a point $l \in B$ by L. For each point $P \in \Gamma$, let $\mathcal{C}_\mathcal{P} = \{l \in B \mid P \in L\}$ and let $C_P = \cup_{l \in \mathcal{C}_P} L \subset X \cap T_P(X)$. Note that the dimension of $\mathcal{C}_\mathcal{P}$ is equal to $N - 3$ (since $\dim(B) = N - 2$ and $X \neq T_P(X)$), so C_P is a hypersurface in $X \cap T_P(X)$. Let P be a general point of Γ. If $C_P = X \cap T_P(X)$, the point P is a star point, hence X has infinitely star points. This is in contradiction with Theorem 4.2. Now assume $C_P \neq X \cap T_P(X)$. So $T_P(X) \cap X$ has an irreducible component D_P of dimension $N - 2$ different from C_P. For each point $Q \in C_P \cap D_P$, we have $T_Q(X) = T_P(X)$ since Q is singular in $X \cap T_P(X)$. This is in contradiction with [19, Corollary 2.8], since $C_P \cap D_P$ has dimension at least one. $\qquad\square$

5 Configurations of star points

In Section 3 of our paper [3], we obtain a lower bound on the dimension of the configuration space for total inflection points on smooth plane curves. This was obtained by describing that configuration space as an intersection of two natural sections of some natural smooth morphism. Now we generalize this to the case of star points. We use Notation 2.8 and introduce some further notations.

Notation 5.1. The set $\mathcal{P}_d^{e,0}$ consists of e-tuples $(\Pi_i, P_i, C_i)_{i=1}^e$ from $\mathcal{P}_d$ satisfying $P_i \notin \Pi_j$ for $i \neq j$. We write $\mathcal{L}_i$ to denote the associated i-tuple $(\Pi_j, P_j, C_j)_{j=1}^i$ for $1 \leq i \leq e$.

The set $\mathcal{V}_{d,e} \subset \mathcal{P}_d^{e,0}$ is determined by the condition that $(\Pi_i, P_i, C_i)_{i=1}^e \in \mathcal{P}_d^{e,0}$ belongs to $\mathcal{V}_{d,e}$ if and only if there exists an irreducible hypersurface X of degree d in $\mathbb{P}^N$ such that $X \cap \Pi_i = C_i$ for $1 \leq i \leq e$.

Notation 5.2. Given $\mathcal{L} \in \mathcal{P}_d^{e,0}$, we write $\Pi_{i,j} = \Pi_i \cap \Pi_j$ for $1 \leq i < j \leq e$. Associated to $\mathcal{L}$, we introduce linear systems g'_i on Π_i for $1 \leq i \leq e$ as follows. In case $i \leq d + 1$, then $g'_i = \Pi_{1,i} + \ldots + \Pi_{i-1,i} + \mathbb{P}_{\Pi_i, d-i+1}$ and g'_i is empty in case $i > d + 1$.

Let $\mathcal{G}^e$ be the space of pairs $(g, \mathcal{L})$ with $\mathcal{L} \in \mathcal{P}_d^{e,0}$ and g is an $(e-1)$-tuple $(g_2, \ldots, g_e)$ of linear systems g_i on Π_i as follows. In case $i \leq d + 1$, then g_i is a linear subsystem of $\mathbb{P}_{\Pi_i, d}$ of dimension $\binom{N+d-i}{N-1}$ containing g'_i and in case $i > d + 1$, g_i has dimension 0 (i.e. it consists of a unique effective divisor). We write $\tau : \mathcal{G}^e \to \mathcal{P}_d^{e,0}$ to denote the natural projection.

14

From the previous definition, it follows that τ is a smooth morphism of relative dimension

$$f = \sum_{i=2}^{e} \left[\binom{N+d-1}{N-1} - \binom{N+d-i}{N-1} - 1 \right]$$

in case $e \leq d+1$ and of relative dimension

$$f = \sum_{i=2}^{d+1} \left[\binom{N+d-1}{N-1} - \binom{N+d-i}{N-1} - 1 \right] + (e-d-1) \left[\binom{N+d-1}{N-1} - 1 \right]$$

in case $e > d+1$.

In the proof of the next theorem we are going to use the following definition.

Definition 5.3. For a point P on a hyperplane $\Pi \subset \mathbb{P}^N$ and an integer $k \geq 1$, we write $k(P \in \Pi)$ to denote the fat point on Π located at P of multiplicity k. So, $k(P \in \Pi)$ is the 0-dimensional subscheme of Π of length $\binom{N+k-2}{N-1}$ with support P and locally defined at P by the ideal $\mathcal{M}_{P,\Pi}^k$.

Theorem 5.4. *There exist two sections S_1, S_2 of τ such that $\mathcal{V}_{d,e} = \tau(S_1 \cap S_2)$.*

Corollary 5.5. *Each irreducible component of $\mathcal{V}_{d,e}$ has codimension at least f inside $\mathcal{P}_d^{e,0}$.*

Proof of theorem 5.4. For $\mathcal{L} \in \mathcal{P}_d^{e,0}$, we define $S_1(\mathcal{L}) = g$ as follows. In case $2 \leq i \leq d+1$, $g_i = \langle g_i', C_i \rangle$ and in case $i > d+1$, then $g_i = \{C_i\}$.

Next we construct the section $S_2(\mathcal{L}) = h$. Consider the surjective map

$$q_2 : V_d(\mathcal{L}_1) \to V_{\Pi_2,d}(\mathcal{L}_1).$$

We know that $\ker(q_1) = \pi_1 \pi_2 V_{d-2}$ (where π_i is the equation of $\Pi_i \subset \mathbb{P}^N$).

Take $h_2 = \mathbb{P}_{\Pi_2,d}(\mathcal{L}_1)$. We know $V_{\Pi_2,d}(\mathcal{L}_1)$ contains $\pi_{1,2}V_{\Pi_2,d-1}$ (where $\pi_{i,j}$ is the equation of $\Pi_{i,j} \subset \Pi_j$) and $\dim(h_2) = \binom{d+N-2}{N-1}$. From the restriction map $\mathcal{O}_{\Pi_2}(d) \to \mathcal{O}_{d(P_2 \in \Pi_2)}(d)$, we obtain a homomorphism

$$V_{\Pi_2,d}(\mathcal{L}_1) \to \Gamma(\mathcal{O}_{d(P_2 \in \Pi_2)}(d)).$$

The image of $\pi_{1,2}V_{\Pi_2,d-1}$ is equal to $\Gamma(\mathcal{O}_{d(P_2 \in \Pi_2)}(d))$, hence the kernel of this map is a 1-dimensional vector space $\langle s_2 \rangle$. Let $V_d'(\mathcal{L}_2) = q^{-1}(\langle s_2 \rangle)$; it is a subspace of V_d containing $\pi_1 \pi_2 V_{d-2}$. Moreover, the associated linear system $\mathbb{P}(V_d'(\mathcal{L}_2))$ has dimension $\binom{N+d-2}{N}$ and it induces a unique divisor on Π_2 with multiplicity at least d at P_2.

Let $2 < i \leq e$ with $i \leq d+1$ and assume h_j is constructed for $2 \leq j \leq i-1$ and assume $V_d'(\mathcal{L}_j) \subset V_d$ is constructed such that $V_d'(\mathcal{L}_j)$ contains $\pi_1 \cdots \pi_j V_{d-j}$, such that $V_d'(\mathcal{L}_j)$ is contained in $V_d'(\mathcal{L}_{j-1})$ (with $V_d'(\mathcal{L}_1) = V_d(\mathcal{L}_1)$) and the associated linear system $\mathbb{P}(V_d'(\mathcal{L}_j))$ has dimension $\binom{N+d-j}{N}$. Assume also that it induces a unique divisor on Π_j with multiplicity at least $d-j+2$ at P_j. The restriction of forms of degree d to Π_i gives rise to a map

$$q_i : V_d'(\mathcal{L}_{i-1}) \to V_{\Pi_i,d} = \Gamma(\Pi_i, \mathcal{O}_{\Pi_i}(d)).$$

15

Assume $s \in \ker(q_i)$, i.e. $\Pi_i \subset Z(s)$. Since $s \in V_d(\mathcal{L}_1)$, we obtain $Z(s) \cap \Pi_1$ contains $C_1 \cup \Pi_{1,i}$. Since $P_1 \notin \Pi_{1,i}$, we find $\Pi_1 \subset Z(s)$. Let $1 < i_0 \leq i - 1$ and assume $\Pi_1 \cup \ldots \cup \Pi_{i_0-1} \subset Z(s)$, then $Z(s) \cap \Pi_{i_0}$ contains

$$\Pi_{1,i_0} \cup \ldots \cup \Pi_{i_0-1,i_0} \cup \Pi_{i_0,i}.$$

If $Z(s)$ does not contain Π_{i_0}, then $Z(s) \cap \Pi_{i_0}$ is a divisor in Π_{i_0} having multiplicity at most $d - i_0$ at P_{i_0}. But $s \in V'_d(\mathcal{L}_{i_0})$, hence we obtain a contradiction and therefore $Z(s)$ contains Π_{i_0}. In this way we find

$$\ker(q_i) = \pi_1 \cdots \pi_i V_{d-i}$$

(hence this kernel is empty in case $i = d + 1$). This implies

$$\dim(\operatorname{im} q_i) = \binom{N+d-i+1}{N} + 1 - \binom{N+d-i}{N} = \binom{N+d-i}{N-1} + 1.$$

Since $\pi_1 \cdots \pi_{i-1} V_{d-i+1} \subset V'_d(\mathcal{L}_{i-1})$, one has

$$\pi_{1,i} \cdots \pi_{i-1,i} V_{\Pi_i, d-i+1} \subset \operatorname{im}(q_i),$$

so we take $h_i = \mathbb{P}(\operatorname{im} q_i)$.

As before, one concludes that the restriction map

$$\operatorname{im}(q_i) \to \Gamma(\mathcal{O}_{(d-i+2)(P_i \in \Pi_i)})$$

is surjective, hence it has a one-dimensional kernel $\langle s_i \rangle$. Let $V'_d(\mathcal{L}_i) = q_i^{-1}(\langle s_i \rangle)$, then the inclusion $V'_d(\mathcal{L}_i) \subset V'_d(\mathcal{L}_{i-1})$ holds, $V'_d(\mathcal{L}_i)$ contains $\pi_1 \cdots \pi_i V_{d-i}$, the associated linear system $\mathbb{P}(V'_d(\mathcal{L}_i))$ has dimension $\binom{N+d-i}{N}$ and it induces a unique divisor on Π_i with multiplicity at least $d - i + 2$ at P_i.

In case $i = d + 1$ (hence $e \geq d + 1$), then $\mathbb{P}(V'_d(\mathcal{L}_{d+1}))$ consists of a unique divisor D. If D contains Π_i for some $i < d + 1$ then as before one proves D contains $\Pi_1, \ldots, \Pi_{d+1}$, hence a contradiction. So for $i > d + 1$, one takes $h_i = \{D\}$.

Clearly $S_1(\mathcal{L}) = S_2(\mathcal{L})$ if and only if $g_i = h_i$ for $2 \leq i \leq e$. The equality $h_2 = g_2$ is equivalent to $C_2 \in \mathbb{P}_{\Pi_2, d}(\mathcal{L}_1)$, which is equivalent to $\mathcal{L}_2 \in \mathcal{V}_{d,2}$ by Theorem 2.10. In this case, we also have $V'_d(\mathcal{L}_2) = V_d(\mathcal{L}_2)$.

Let $3 \leq i \leq e$ and assume that $h_j = g_j$ for $2 \leq j \leq i - 1$ is equivalent to $\mathcal{L}_{i-1} \in \mathcal{V}_{d,i-1}$ and in this case, $V'_d(\mathcal{L}_{i-1}) = V_d(\mathcal{L}_{i-1})$. Assume $h_j = g_j$ for $2 \leq j \leq i$. Hence, $\mathcal{L}_{i-1} \in \mathcal{V}_{d,i-1}$ and the image of q_i is equal to $V_{\Pi_i, d}(\mathcal{L}_{i-1})$. From the previous arguments, it follows that $\mathbb{P}(\operatorname{im}(q_i)) = h_i$ contains a unique divisor having multiplicity at least $\max\{d - i + 2, 0\}$ at P_i, hence $h_i = g_i$ is equivalent to $C_i \in \mathbb{P}(\operatorname{im} q_i) = \mathbb{P}_{\Pi_i, d}(\mathcal{L}_i)$. This is equivalent to $\mathcal{L}_i \in \mathcal{V}_{d,i}$ and by construction $V'_d(\mathcal{L}_i) = V_d(\mathcal{L}_i)$ in that case.

By induction on i, we conclude $S_1(\mathcal{L}) = S_2(\mathcal{L})$ if and only if $\mathcal{L} \in \mathcal{V}_{d,e}$. $\square$

Remark 5.6. Take $N = 3$, hence $\dim(\mathcal{P}_d^{e,0}) = e(d+5)$.

In case $e = 2$, we find $f = d$, hence $\dim(\mathcal{V}_{d,2}) \geq d + 10$. Combining this with Theorem 2.10, the space of surfaces in $\mathbb{P}^3$ having at least 2 star points not contained in a line inside the surface has dimension at least $d + 10 + \binom{d+1}{3}$. In case $d = 3$, we find this space has dimension at least 17. In his thesis [10] (see also [11]), Nguyen found exactly two components both having dimension 17.

In case $e = 3$, we find $f = 3d$, hence $\dim(\mathcal{V}_{d,3}) \geq 15$. In case $\mathcal{V}_{d,3}$ would have a component of dimension exactly 15, it would be an orbit under the action of $\mathrm{Aut}(\mathbb{P}^3)$. Combining this description with Theorem 2.10, the space of hypersurfaces with three star points has dimension at least $15 + \binom{d}{3}$. In case $d = 3$, we find this space has dimension at least 16. In his work, Nguyen found a unique component of dimension 17. This component comes from the fact that a cubic surface X in $\mathbb{P}^3$ having two star points P_1 and P_2, such that the line connecting those points is not contained in X, has at least three star points (see Theorem 3.4). It follows that the space of cubic surfaces with at least three star points not on a line has only irreducible components of dimension 16 (but those components do parameterize surface having more than three star points because of Theorem 3.4).

The following proposition is a generalization of a result on total inflection points on smooth plane curves proved in [16] to the case of star points. It shows that the case of three star points (see Section 7) and collinear star points (see Section 3) are the most basic cases. The proof given in [16] relies on a very general formulated algebraic statement, that can also be applied in the situation of star points. Here we give a more direct and more geometric proof based on Theorem 2.10.

Proposition 5.7. *Let* $\mathcal{L} = (\Pi_i, P_i, C_i)_{i=1}^e \in \mathcal{P}_d^{e,0}$. *Assume that*

(i) *for all* $1 \leq i_1 < i_2 < i_3 \leq e$, *one has* $(\Pi_{i_j}, P_{i_j}, C_{i_j})_{j=1}^3 \in \mathcal{V}_{d,3}$;

(ii) *for all* $1 \leq i_1 < i_2 < \cdots < i_m \leq e$ *with* $\dim(\Pi_{i_1} \cap \Pi_{i_2} \cap \cdots \cap \Pi_{i_m}) = N - 2$, *one has* $(\Pi_{i_j}, P_{i_j}, C_{i_j})_{j=1}^m \in \mathcal{V}_{d,m}$;

then $\mathcal{L} \in \mathcal{V}_{d,e}$.

Proof. Of course we may assume $e > 3$. Assume both conditions hold and assume the conclusion of the proposition holds for $e - 1$ instead of e.

In case $\Pi_{i,e} = \Pi_{j,e}$ for all $1 \leq i < j \leq e - 1$, then the second assumption (applied to $i_j = j$ for $1 \leq j \leq e$) implies $\mathcal{L} \in \mathcal{V}_{d,e}$.

So we can assume $\Pi_{1,e} \neq \Pi_{2,e}$. Applying the induction hypothesis to $i \in \{1, \cdots, e - 1\}$, we find $X' \in \mathbb{P}_d$ such that $X' \cap \Pi_i = C_i$ for $1 \leq i \leq e - 1$. Let $X' \cap \Pi_e = C'_e$. Let φ'_e (resp. φ_e) be the equation of C'_e (resp. C_e) inside Π_e. For $j \in \{1, 2\}$, consider the linear system of hypersurfaces $X \in \mathbb{P}_d$ such that $X \cap \Pi_i = C_i$ for $i \in \{j, 3, \ldots, e - 1\}$. On Π_e, this linear subsystem of $\mathbb{P}_d$ induces a linear subsystem of $\mathbb{P}_{\Pi_e, d}$. From the induction hypothesis applied to $\{j, 3, \ldots, e\}$ this linear subsystem of $\mathbb{P}_{\Pi_e, d}$ is equal to

$$\left\langle \Pi_{j,e} + \sum_{i=3}^{e-1} \Pi_{i,e} + \mathbb{P}_{\Pi_e, d-e+2}, C_e \right\rangle$$

17

in case $d - e + 2 \geq 0$; otherwise it is $\{C_e\}$.

Since C'_e belongs to those linear subsystems of $\mathbb{P}_{\Pi_e,d}$, there exists $g_j \in V_{\Pi_e,d-e+2}$ and $c_j \in \mathbb{C}$ such that $\varphi'_e = \pi_{j,e}.\prod_{i=3}^{e-1} \pi_{i,e}.g_j + c_j.\varphi_e$ (here we write $\pi_{i,e}$ to denote the equation of $\Pi_{i,e}$ inside Π_e) in case $d - e + 2 \geq 0$; in case $d - e + 2 < 0$ it already proves $C'_e = C_e$. In case $d - e + 2 \geq 0$, we find

$$\prod_{i=3}^{e-1} \pi_{i,e}.(\pi_{1,e}g_1 - \pi_{2,e}g_2) + (c_2 - c_1)\varphi_e = 0$$

Since $\pi_{i,e}$ (with $i \in \{3, \ldots, e-1\}$) cannot divide φ_e, it follows that $c_1 = c_2$ and $\pi_{1,e}g_1 = \pi_{2,e}g_2$. So we have that $g_1 = \pi_{2,e}g'_1$ (here we use $\Pi_{1,e} \neq \Pi_{2,e}$), hence $\varphi'_e = \prod_{i=1}^{e-1} \pi_{i,e}g'_1 + c_1\varphi_e$.

Since $\pi_{1,e}$ cannot divide φ'_e, it follows that $c_1 \neq 0$, thus

$$C_e \in \left\langle \sum_{i=1}^{e-1} \Pi_{i,e} + \mathbb{P}_{\Pi_e,d-e+1}, C'_e \right\rangle \subset \mathbb{P}_{\Pi_e,d},$$

hence C_e belongs to the linear system on Π_e induced by the linear subsystem of $\mathbb{P}_d$ of hypersurfaces X satisfying $C_i \subset X$ for $1 \leq i \leq e-1$. From Theorem 2.10, it follows that $\mathcal{L} \in \mathcal{V}_{d,e}$. $\square$

If follows from the considerations in Section 7.1.3 that in case

$$\dim(\Pi_{i_1} \cap \Pi_{i_2} \cap \cdots \cap \Pi_{i_m}) = N - 2,$$

the points $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ are collinear. This situation can also be handled using the arguments from Theorem 2.10.

6 Special case : hypersurfaces with two star points

In this Section, we will prove the following result.

Theorem 6.1. *Consider the set $\mathcal{S}$ of configurations $\mathcal{L} \in (\mathcal{P}_d)^2$ that are suited for degree $d \geq 3$ in $\mathbb{P}^N$. Then $\mathcal{S}$ has two irreducible components. The set $\mathcal{V}_{d,2}$ is one of these components and it has the expected dimension.*

First assume $\mathcal{L} = (\Pi_i, P_i, C_i)_{i=1}^2$ is a point in $\mathcal{V}_{d,2}$, so $P_1 \notin \Pi_2$ and $P_2 \notin \Pi_1$. We can introduce coordinates $(X_0 : \ldots : X_n)$ on $\mathbb{P}^N$ such that Π_1 has equation $X_1 = 0$, $P_1 = (1 : 0 : 0 : \ldots : 0)$, Π_2 has equation $X_0 = 0$ and $P_2 = (0 : 1 : 0 : \ldots : 0)$. Let $X = Z(f)$ be a hypersurface of degree d in $\mathbb{P}_d(\mathcal{L})$. We can write $f(X_0, \ldots, X_N)$ as

$$g_{01}(X_0, \ldots, X_N)X_0X_1 + g_0(X_0, X_2, \ldots, X_N)X_0$$
$$+ g_1(X_1, X_2, \ldots, X_N)X_1 + g(X_2, \ldots, X_N).$$

18

Since the equation $g_0(X_0, X_2, \ldots, X_N)X_0 + g(X_2, \ldots, X_N) = 0$ of the cone C_1 in Π_1 has to be independent of the variable X_0, we have $g_0(X_0, X_2, \ldots, X_N) \equiv 0$. Analogously, we get that $g_1(X_1, X_2, \ldots, X_N) \equiv 0$, since $C_2 \subset \Pi_2$ is given by $g_1(X_1, X_2, \ldots, X_N)X_1 + g(X_2, \ldots, X_N) = 0$. So, we conclude that

$$f = g_{01}(X_0, \ldots, X_N)X_0X_1 + g(X_2, \ldots, X_N). \tag{1}$$

The cones $C_1 \subset \Pi_1$ and $C_2 \subset \Pi_2$ are both determined by $g(X_2, \ldots, X_N) = 0$. Thus the dimension of $\mathbb{P}_d(\mathcal{L})$ equals $\binom{N+d-2}{N}$ and $\mathcal{V}_{d,2}$ is irreducible and has dimension

$$2N + 2(N-1) + \binom{N+d-2}{N-2} - 1,$$

which is the expected dimension.

Now assume $\mathcal{L} = (\Pi_i, P_i, C_i)_{i=1}^2 \notin \mathcal{P}_d^{2,0}$, so $P_1 \in \Pi_2$ or $P_2 \in \Pi_1$. If $P_1 \in \Pi_2$, we have that $\langle P_1, P_2 \rangle \subset C_2$, hence also $\langle P_1, P_2 \rangle \subset C_1$ or $P_2 \in \Pi_1$. We may choose coordinates $(X_0 : \ldots : X_N)$ on $\mathbb{P}^N$ such that $P_1 = (1 : 0 : 0 : \ldots : 0)$, $P_2 = (0 : 1 : 0 : \ldots : 0)$, $\Pi_1 : X_2 = 0$ and $\Pi_2 : X_3 = 0$. Let $X = Z(f)$ be a hypersurface in $\mathbb{P}_d(\mathcal{L})$. We can write f as

$$g_{23}(X_0, \ldots, X_N)X_2X_3 + g_2(X_0, X_1, X_2, X_4, \ldots, X_N)X_2$$
$$+ g_3(X_0, X_1, X_3, X_4, \ldots, X_N)X_3 + g(X_0, X_1, X_4, \ldots, X_N).$$

Since $C_1 \subset \Pi_1$ is given by

$$g_3(X_0, X_1, X_3, X_4, \ldots, X_N)X_3 + g(X_0, X_1, X_4, \ldots, X_N) = 0,$$

we have that g_3 and g are independent of X_0. Analogously, by considering $C_2 \subset \Pi_2$, we get that g_2 and g are independent of X_1. We conclude that f can be written as

$$g_{23}(X_0, \ldots, X_N)X_2X_3 + g_2(X_0, X_2, X_4, \ldots, X_N)X_2$$
$$+ g_3(X_1, X_3, X_4, \ldots, X_N)X_3 + g(X_4, \ldots, X_N) \tag{2}$$

If $N = 3$, we have $g \equiv 0$ and

$$f = g_{23}(X_0, X_1, X_2, X_3)X_2X_3 + g_2(X_0, X_2)X_2 + g_3(X_1, X_3)X_3.$$

The cones $C_1 \subset \Pi_1$ and $C_2 \subset \Pi_2$ are determined by respectively $g_3 = 0$ and $g_2 = 0$, so the dimension of $\mathbb{P}_d(\mathcal{L})$ equals $\binom{d+1}{3} + 1$ and the dimension of the irreducible locus in $\mathcal{P}_d^2$ is $2.3 + 2.1 + 2(d-1) = 2(d+3)$.

If $N > 3$, the dimension of $\mathbb{P}_d(\mathcal{L})$ is $\binom{N+d-2}{N}$ and the irreducible locus in $\mathcal{P}_d^2$ has dimension equal to

$$2N + 2(N-2) + \binom{N+d-4}{N-4} + 2\binom{N+d-3}{N-2} - 1.$$

7 Special case : hypersurfaces with three star points

7.1 Components of $\mathcal{V}_{d,3}$

In this section (including Subsections 7.1.1-7.1.4), we will prove the following theorem.

Theorem 7.1. *The configuration space $\mathcal{V}_{d,3}$ (where $d \geq 3$) has $2d-2$ irreducible components, of which $\phi(d) + \phi(d-1)$ for $N = 3$ and $\phi(d)$ for $N > 3$ are of the expected dimension.*

Let $\mathcal{L} = (\Pi_i, P_i, C_i)_{i=1}^3$ be an element of $\mathcal{V}_{d,3}$, so $P_i \notin \Pi_j$ for $i \neq j$. We can choose coordinates $(X_0 : \ldots : X_N)$ on $\mathbb{P}^N$ such that $P_1 = (1 : 0 : 0 : \ldots : 0)$, Π_1 has equation $X_1 = 0$, $P_2 = (0 : 1 : 0 : \ldots : 0)$ and Π_2 has equation $X_0 = 0$. Let $X = Z(f)$ be a hypersurface in $\mathbb{P}_d(\mathcal{L})$. Using Section 6, we have that f is of the form $g_{01}(X_0, \ldots, X_N)X_0X_1 + g(X_2, \ldots, X_N)$.

7.1.1 Case I: $P_3 \notin \langle P_1, P_2 \rangle$ and $\Pi_1 \cap \Pi_2 \not\subset \Pi_3$

In this case, we may assume that $P_3 = (a : b : a + b : 0 : \ldots : 0)$ and Π_3 has equation $X_2 = X_0 + X_1$. Note that $a \neq 0$, $b \neq 0$ and $a + b \neq 0$ since respectively $P_3 \notin \Pi_2$, $P_3 \notin \Pi_1$ and $P_3 \notin \langle P_1, P_2 \rangle$. So we may take $a = -1$ and $b = t$ with $t \neq 1$ and $t \neq 0$.

Assume for simplicity that $N = 3$ and consider the coordinate transformation defined by

$$
\begin{cases}
Y_0 & = & tX_0 + X_1 \\
Y_1 & = & X_0 + X_1 - X_2 \\
Y_2 & = & X_2 \\
Y_3 & = & X_3
\end{cases}
\iff
\begin{cases}
X_0 & = & \frac{1}{t-1}(Y_0 - Y_1 - Y_2) \\
X_1 & = & \frac{1}{t-1}(-Y_0 + tY_1 + tY_2) \\
X_2 & = & Y_2 \\
X_3 & = & Y_3
\end{cases},
$$

so for the new coordinate system, we have $P_3 = (0 : 0 : 1 : 0)$ and $\Pi_3 : Y_1 = 0$. The equation of X becomes

$$
\frac{(Y_0 - Y_1 - Y_2)(-Y_0 + tY_1 + tY_2)}{(t-1)^2} g_{01}\left(\frac{Y_0 - Y_1 - Y_2}{t-1}, \frac{-Y_0 + tY_1 + tY_2}{t-1}, Y_2, Y_3\right)
$$
$$
+ g(Y_2, Y_3) = 0, \quad (3)
$$

so the cone $C_3 \subset \Pi_3$ is given by

$$
\frac{(Y_0 - Y_2)(-Y_0 + tY_2)}{(t-1)^2} g_{01}\left(\frac{Y_0 - Y_2}{t-1}, \frac{-Y_0 + tY_2}{t-1}, Y_2, Y_3\right) + g(Y_2, Y_3) = 0. \quad (4)
$$

The left hand side of (4) should be independent of the variable Y_2. If we write

$$
g_{01}\left(\frac{Y_0 - Y_2}{t-1}, \frac{-Y_0 + tY_2}{t-1}, Y_2, Y_3\right) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d-2}} c_{i,j} Y_0^i Y_2^j Y_3^{d-i-j-2}
$$

and

$$g(Y_2, Y_3) = \sum_{0 \le j \le d} d_j Y_2^j Y_3^{d-j},$$

the equation (4) is independent of Y_2 if and only if

$$\begin{cases}
-tc_{i,j-2} + (t+1)c_{i-1,j-1} - c_{i-2,j} = 0 & \text{for } i,j \ge 2 \\
(t+1)c_{i-1,0} - c_{i-2,1} = 0 & \text{for } i \ge 2 \\
-tc_{1,j-2} + (t+1)c_{0,j-1} = 0 & \text{for } j \ge 2 \\
\frac{-t}{(t-1)^2}c_{0,j-2} + d_j = 0 & \text{for } j \ge 2 \\
(t-1)c_{0,0} = d_1 = 0
\end{cases} \tag{5}$$

So for each $j \in \{2, \ldots, d\}$, the coefficients $c_{j-2,0}, c_{j-3,1}, \ldots, c_{0,j-2}$ satisfy a system Σ_j of linear equations, for which the coefficient matrix is equal to the following tridiagonal matrix

$$M_j = \begin{bmatrix}
t+1 & -1 & \ddots & 0 & 0 \\
-t & t+1 & \ddots & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & t+1 & -1 \\
0 & 0 & \ddots & -t & t+1
\end{bmatrix} \in \mathbb{C}^{(j-1)\times(j-1)}.$$

The determinant of M_j equals

$$\frac{t^j - 1}{t - 1} = 1 + t + t^2 + \ldots + t^{j-1},$$

so Σ_j has a non-zero solution if and only if $t^j = 1$. In this case, the solutions of Σ_j are given by

$$\begin{aligned}
(c_{j-2,0}, c_{j-3,1}, \ldots, c_{0,j-2}) &= A_j \cdot (1, 1+t, \ldots, 1+t+t^2+\ldots+t^{j-1}) \\
&= A_j\left(\tfrac{t-1}{t-1}, \tfrac{t^2-1}{t-1}, \ldots, \tfrac{t^{j-1}-1}{t-1}\right),
\end{aligned}$$

where $A_j \in \mathbb{C}$, hence

$$\begin{aligned}
\sum_{k=0}^{j-2} c_{j-k-2,k} Y_0^{j-k-2} Y_2^k &= A_j \cdot \left(\sum_{k=0}^{j-2} \tfrac{t^{k+1}-1}{t-1} Y_0^{j-k-2} Y_2^k\right) \\
&= \tfrac{A_j}{t-1}\left(t \cdot \tfrac{Y_0^{j-1}-(tY_2)^{j-1}}{Y_0 - tY_2} - \tfrac{Y_0^{j-1}-Y_2^{j-1}}{Y_0 - Y_2}\right) \\
&= A_j \cdot \tfrac{Y_0^j - Y_2^j}{(Y_0-Y_2)(Y_0-tY_2)}.
\end{aligned}$$

So we get

$$g_{01}\left(\frac{Y_0 - Y_2}{t-1}, \frac{-Y_0 + tY_2}{t-1}, Y_2, Y_3\right) = \sum_{\substack{0 < j \le d \\ t^j = 1}} A_j \frac{(Y_0^j - Y_2^j)Y_3^{d-j}}{(Y_0 - Y_2)(Y_0 - tY_2)}.$$

21

This implies that $g_{01}(X_0, X_1, X_2, X_3)$ is of the form

$$\sum_{\substack{0 < j \leq d \\ t^j = 1}} A_j \frac{((tX_0 + X_1)^j - X_2^j) X_3^{d-j}}{(tX_0 + X_1 - X_2)(tX_0 + X_1 - tX_2)} + (X_0 + X_1 - X_2) g_{012},$$

where $g_{012} \in \mathbb{C}[X_0, X_1, X_2, X_3]_{d-3}$.

Since

$$c_{0,j-2} = A_j . \frac{t^{j-1} - 1}{t - 1} = \frac{-A_j}{t},$$

from (5) follows that $d_j = \frac{-A_j}{(t-1)^2}$, thus

$$g(X_2, X_3) = \frac{-1}{(t-1)^2} \sum_{\substack{0 \leq j \leq d \\ t^j = 1}} A_j X_2^j X_3^{d-j},$$

where we take $A_0 = -(t-1)^2 d_0$.

If $t^d \neq 1$ and $t^{d-1} \neq 1$, we see that f belongs to the ideal $I := \langle X_0 X_1, X_3^2 \rangle$, hence $P(0:0:1:0) \in \mathrm{Sing}(X)$, a contradiction. So we have either $t^d = 1$ or $t^{d-1} = 1$. In these cases, for general $A_j \in \mathbb{C}$ and $g_{012} \in \mathbb{C}[X_0, X_1, X_2, X_3]_{d-3}$, the surface $X = Z(f)$ will be smooth.

If $t^d = 1$ or $t^{d-1} = 1$, denote the component of $\mathcal{V}_{d,3}$ corresponding to the root of unity $t \neq 1$ by $\mathcal{V}_t$ and denote the order of t in $\mathbb{C}^\star$ by $\theta(t)$. If we fix $\mathcal{L} \in \mathcal{V}_t$, the values of the numbers A_j are fixed (up to a scalar), but g_{012} can vary, so the dimension of $\mathbb{P}_d(\mathcal{L})$ equals $\binom{d}{3}$. In case $t^d = 1$, the dimension of $\mathcal{V}_t$ equals

$$3.3 + 2.2 + 1 + \left(\frac{d}{\theta(t)} + 1 \right) - 1 = 14 + \frac{d}{\theta(t)},$$

since we can choose arbitrarily $\Pi_1, \Pi_2, \Pi_3 \subset \mathbb{P}^3$, $P_1 \in \Pi_1$, $P_2 \in \Pi_2$, P_3 on a certain line in Π_3 and $A_0, A_{\theta(t)}, \ldots, A_d$ (up to a scalar). Analogously, we can see that in case $t^{d-1} = 1$, the dimension of $\mathcal{V}_t$ equals $14 + \frac{d-1}{\theta(t)}$. Note that in both cases, the dimension of $\mathcal{V}_t$ is dependent of $\theta(t)$ and it is equal to the expected dimension 15 if and only if $\theta(t)$ is maximal (i.e. $\theta(t) \in \{d-1, d\}$).

These results can easily be generalized to $N > 3$. Indeed, the defining polynomial f of X becomes

$$X_0 X_1 . \sum_{0 < j \leq d; \ t^j = 1} A_j(X_3, \ldots, X_N) \frac{(tX_0 + X_1)^j - X_2^j}{(tX_0 + X_1 - X_2)(tX_0 + X_1 - tX_2)} +$$

$$X_0 X_1 (X_0 + X_1 - X_2) g_{012}(X_0, \ldots, X_N) - \sum_{0 \leq j \leq d; \ t^j = 1} \frac{A_j(X_3, \ldots, X_N) X_2^j}{(t-1)^2},$$

where A_j is a polynomial of degree $d - j$.

If $t^d \neq 1$ and $t^{d-1} \neq 1$, we see that f belongs to the ideal I generated by $X_0 X_1$ and $X_i X_j$ with $3 \leq i \leq j \leq N$, thus $P(0:0:1:0:\ldots:0) \in \text{Sing}(X)$, a contradiction. So we have either $t^d = 1$ or $t^{d-1} = 1$. In this case, let $\theta(t)$ be the order of t in $\mathbb{C}^\star$ and write $\mathcal{V}_t$ to denote the component corresponding to $t \neq 1$. The dimension of $\mathbb{P}_d(\mathcal{L})$ is $\binom{N+d-3}{N}$ and $\mathcal{V}_t$ has dimension equal to

$$3N + 2(N-1) + (N-2) + \sum_{0 \leq j \leq d;\ t^j = 1} \binom{N+d-3-j}{N-3} - 1$$

$$= 6N + \sum_{0 \leq j \leq d;\ t^j = 1} \binom{N+d-3-j}{N-3} - 5.$$

The component $\mathcal{V}_t$ has the expected dimension, which is equal to

$$6N + \binom{N+d-3}{N-3} - 4,$$

if and only if t is a primitive dth root of unity (i.e. $\theta(t) = d$).

7.1.2 Case II: $P_3 \in \langle P_1, P_2 \rangle$ and $\Pi_1 \cap \Pi_2 \not\subset \Pi_3$

We may assume that $P_3 = (1:-1:0:\ldots:0)$ and $\Pi_3 : X_2 = X_0 + X_1$. Consider the coordinate transformation defined by

$$\begin{cases} Y_0 &= X_0 \\ Y_1 &= X_0 + X_1 \\ Y_2 &= X_2 - X_0 - X_1 \\ Y_i &= X_i \quad (\forall i \geq 3) \end{cases} \iff \begin{cases} X_0 &= Y_0 \\ X_1 &= Y_1 - Y_0 \\ X_2 &= Y_1 + Y_2 \\ X_i &= Y_i \quad (\forall i \geq 3) \end{cases},$$

hence in the new system, we have $P_3 = (1:0:0:\ldots:0)$, $\Pi_3 : Y_2 = 0$ and X is defined by

$$Y_0(Y_1 - Y_0)g_{01}(Y_0, Y_1 - Y_0, Y_1 + Y_2, Y_3, \ldots, Y_N) + g(Y_1 + Y_2, Y_3, \ldots, Y_N) = 0.$$

The cone $C_3 \subset \Pi_3$ is given by

$$Y_0(Y_1 - Y_0)g_{01}(Y_0, Y_1 - Y_0, Y_1, Y_3, \ldots, Y_N) + g(Y_1, Y_3, \ldots, Y_N) = 0. \quad (6)$$

Since (6) has to be independent of the variable Y_0, we get that $g_{01}(Y_0, Y_1 - Y_0, Y_1, Y_3, \ldots, Y_N) \equiv 0$, thus $g_{01}(X_0, \ldots, X_N)$ is divisible by $X_2 - X_0 - X_1$ and f is of the form

$$X_0 X_1(X_2 - X_0 - X_1)g_{012}(X_0, \ldots, X_N) + g(X_2, \ldots, X_N).$$

This case gives rise to a component of $\mathcal{V}_{d,3}$ not described in Case I, which we will denote by $\mathcal{V}_1$. If we fix $\mathcal{L}$, the polynomial g is fixed, but g_{012} can vary. So the dimension of $\mathbb{P}_d(\mathcal{L})$ is equal to $\binom{N+d-3}{N}$ and the dimension of $\mathcal{V}_1$ is

$$3N + 2(N-1) + \binom{N+d-2}{N-2} - 1.$$

23

7.1.3 Case III: $P_3 \notin \langle P_1, P_2 \rangle$ and $\Pi_1 \cap \Pi_2 \subset \Pi_3$

We can take the coordinates on $\mathbb{P}^N$ so that $P_3 = (0 : 0 : 1 : 0 : \ldots : 0)$ and $\Pi_3 : X_0 = X_1$. The cone C_3 is defined by $X_1 - X_0 = 0$ and

$$X_0^2 g_{01}(X_0, X_0, X_2, X_3, \ldots, X_N) + g(X_2, \ldots, X_N) = 0. \tag{7}$$

Since (7) has to be independent of X_2, we get that g is independent of X_2 and $P_2 \in \operatorname{Sing}(X)$, a contradiction. So there are no elements $\mathcal{L} \in \mathcal{V}_{d,3}$ of this form.

7.1.4 Case IV: $P_3 \in \langle P_1, P_2 \rangle$ and $\Pi_1 \cap \Pi_2 \subset \Pi_3$

Assume that $P_3 = (1 : 1 : 0 : \ldots : 0)$ and $\Pi_3 : X_0 = X_1$. Consider the following coordinate transformation

$$\begin{cases} Y_0 &= X_0 \\ Y_1 &= X_1 - X_0 \\ Y_i &= X_i \quad (\forall i \geq 2) \end{cases} \iff \begin{cases} X_0 &= Y_0 \\ X_1 &= Y_0 + Y_1 \\ X_i &= Y_i \quad (\forall i \geq 2) \end{cases}.$$

For the new coordinates, we have $P_3 = (1 : 0 : \ldots : 0)$, $\Pi_3 : Y_1 = 0$ and X is given by

$$Y_0(Y_0 + Y_1)g_{01}(Y_0, Y_0 + Y_1, Y_2, \ldots, Y_N) + g(Y_2, \ldots, Y_N) = 0.$$

The equation of the cone $C_3 \subset \Pi_3$ is

$$Y_0^2 g_{01}(Y_0, Y_0, Y_2, \ldots, Y_N) + g(Y_2, \ldots, Y_N) = 0. \tag{8}$$

Since (8) has to be independent of Y_0, we get $g_{01}(Y_0, Y_0, Y_2, \ldots, Y_N) \equiv 0$ and so $g_{01}(X_0, \ldots, X_N)$ is divisible by $X_1 - X_0$. Thus, f is of the form

$$X_0 X_1 (X_1 - X_0) g_{01}(X_0, \ldots, X_N) + g(X_2, \ldots, X_N).$$

It is clear that in this case, the elements $\mathcal{L}$ are contained in the component $\mathcal{V}_1$. Moreover, this subset of $\mathcal{V}_1$ has dimension

$$2N + 2(N-1) + 1 + \binom{N+d-2}{N-2} - 1.$$

7.1.5 Conclusion

The set $\mathcal{V}_{d,3}$ has $(d-1) + (d-2) + 1 = 2d - 2$ components, namely $\mathcal{V}_t$ with t a root of the polynomial

$$\frac{(t^d - 1)(t^{d-1} - 1)}{t - 1} \in \mathbb{C}[t].$$

The component $\mathcal{V}_t$ has the expected dimension if and only if $\theta(t) \in \{d-1, d\}$ if $N = 3$ and $\theta(t) = d$ if $N > 3$. So, $\mathcal{V}_{d,3}$ has $\phi(d) + \phi(d-1)$ components of the expected dimension if $N = 3$ and $\phi(d)$ if $N > 3$.

24

Remark 7.2. For smooth cubic surfaces (i.e. $d = N = 3$), the set $\mathcal{V}_{d,3}$ has three components of the expected dimension 15 (namely $\mathcal{V}_{-1}$, $\mathcal{V}_{\omega_3}$ and $\mathcal{V}_{\omega_3^2}$, where $\omega_3 = e^{2\pi i/3}$) and one component $\mathcal{V}_1 \subset \mathcal{V}_{d,3}$ of dimension 16.

Remark 7.3. Consider the Fermat hypersurface $X_{d,N}$ (see Example 2.13). Let i, j, k be different elements in $\{0, \ldots, N\}$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$ such that $\xi_1^d = \xi_2^d = \xi_3^d = -1$. Then the triple of star points

$$(E_{i,j}(\xi_1), E_{i,j}(\xi_2), E_{i,k}(\xi_3)),$$

with $\xi_1 \neq \xi_2$, belongs to the component $\mathcal{V}_t$ with $t = \frac{\xi_1}{\xi_2}$. On the other side, the triple of star points

$$(E_{i,j}(\xi_1), E_{i,k}(\xi_2), E_{j,k}(\xi_3))$$

belongs to $\mathcal{V}_t$ with $t = (\frac{\xi_2}{\xi_1 \xi_3})^2$.

7.2 Components outside $\mathcal{V}_{d,3}$: Intermediate case

We show the following result for the intermediate case.

Theorem 7.4. *Let $\mathcal{V}_{int}$ be the set of configurations $\mathcal{L} \in (\mathcal{P}_d)^3$ that are suited for degree $d \geq 3$ in $\mathbb{P}^N$ and such that $P_1 \notin \Pi_2 \cup \Pi_3$ but $\langle P_2, P_3 \rangle \subset \Pi_2 \cap \Pi_3$. Then $\mathcal{V}_{int}$ is irreducible of dimension*

$$3N + (N-1) + 2(N-2) + \binom{N+d-3}{N-3} + \sum_{k=1}^{d-1} \binom{N+d-k-4}{N-3} - 1.$$

Let $\mathcal{L} \in (\mathcal{P}_d)^3$ with $\mathbb{P}_d(\mathcal{L}) \neq \emptyset$ such that $P_1 \notin \Pi_2 \cup \Pi_3$ and $P_2, P_3 \notin \Pi_1$, but $P_2 \in \Pi_3$. In this case, we have that $\langle P_2, P_3 \rangle \subset \Pi_2 \cap \Pi_3$.

We can choose coordinates $(X_0 : \ldots : X_N)$ such that $P_1 = (1 : 0 : 0 : \ldots : 0)$, Π_1 has equation $X_1 = 0$, $P_2 = (0 : 1 : 0 : \ldots : 0)$, Π_2 has equation $X_0 = 0$, $P_3 = (0 : 1 : 1 : 0 : \ldots : 0)$ and Π_3 has equation $X_0 = X_3$. From Section 6 follows that f is of the form

$$X_0 X_1 g_{01}(X_0, \ldots, X_N) + g(X_2, \ldots, X_N).$$

Consider the following coordinate transformation

$$\begin{cases} Y_0 &= X_0 - X_3 \\ Y_1 &= X_1 - X_2 \\ Y_i &= X_i \quad (\forall i \geq 2) \end{cases} \iff \begin{cases} X_0 &= Y_0 + Y_3 \\ X_1 &= Y_1 + Y_2 \\ X_i &= Y_i \quad (\forall i \geq 2) \end{cases}.$$

In this system, $P_3 = (0 : 0 : 1 : 0 : \ldots : 0)$, $\Pi_3 : Y_0 = 0$ and X has equation

$$(Y_0 + Y_3)(Y_1 + Y_2) g_{01}(Y_0 + Y_3, Y_1 + Y_2, Y_2, Y_3, \ldots, Y_N) + g(Y_2, Y_3, \ldots, Y_N) = 0,$$

so the cone $C_3 \subset \Pi_3$ is given by

$$Y_3(Y_1 + Y_2) g_{01}(Y_3, Y_1 + Y_2, Y_2, Y_3, \ldots, Y_N) + g(Y_2, Y_3, \ldots, Y_N) = 0. \quad (9)$$

For simplicity, we will first assume $N = 3$. If we write

$$g_{01}(Y_3, Y_1 + Y_2, Y_2, Y_3) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d-2}} a_{i,j} Y_1^i Y_2^j Y_3^{d-i-j-2}$$

and $g(Y_2, Y_3) = \sum_{j=0}^{d} b_j Y_2^j Y_3^{d-j}$, we see that the equation (9) of C_3 is independent of the variable Y_2 if and only if

$$\begin{cases} a_{i-1,j} + a_{i,j-1} = 0 & (\forall i > 0, j > 0 \text{ with } i+j \leq d-1) \\ b_j + a_{0,j-1} = 0 & (\forall j \in \{1, \ldots, d-1\}) \\ b_d = 0 \end{cases} \qquad (10)$$

If (10) is satisfied, we have $a_{i,k-i-1} = -(-1)^i b_k$ for all $1 \leq k \leq d-1$, and thus

$$\sum_{i=0}^{k-1} a_{i,k-i-1} Y_1^i Y_2^{k-i-1} = -b_k \sum_{i=0}^{k} (-Y_1)^i Y_2^{k-i-1} = -b_k \frac{Y_2^k - (-Y_1)^k}{Y_2 + Y_1}.$$

It follows

$$g_{01}(Y_3, Y_1 + Y_2, Y_2, Y_3) = -\frac{1}{Y_2 + Y_1} \sum_{k=1}^{d-1} b_k (Y_2^k - (-Y_1)^k) Y_3^{d-k-1},$$

so $g_{01}(X_0, X_1, X_2, X_3)$ is of the form

$$(X_3 - X_0) g_{013}(X_0, \ldots, X_N) - \frac{1}{X_1} \sum_{k=1}^{d-1} b_k (X_2^k - (X_2 - X_1)^k) X_3^{d-k-1},$$

where g_{013} is a polynomial of degree $d - 3$. We conclude that f is of the form

$$X_0 X_1 (X_3 - X_0) g_{013} - X_0 \sum_{k=1}^{d-1} b_k (X_2^k - (X_2 - X_1)^k) X_3^{d-k-1} + \sum_{k=0}^{d-1} b_k X_2^k X_3^{d-k}.$$

So, in this case, for $N = 3$, the dimension of $\mathbb{P}_d(\mathcal{L})$ is $\binom{d}{3}$ and we get a component of the set of surfaces with three star points of dimension

$$3.3 + 2 + 2.1 + d + \binom{d}{3} - 1 = 12 + d + \binom{d}{3}.$$

These results can easily be generalized to general values of $N \geq 3$. Indeed, in this case, f is of the form

$$X_0 X_1 (X_3 - X_0) g_{013} - X_0 \sum_{k=1}^{d-1} B_k (X_2^k - (X_2 - X_1)^k) + X_3 \sum_{k=1}^{d-1} B_k X_2^k + B_0,$$

where the polynomials B_i are homogeneous in the variables $X_3, \ldots, X_N$ of degree d for $i = 0$ and $d - k - 1$ for $k \in \{1, \ldots, d-1\}$. We get a component of the set of hypersurfaces with three star points of dimension

$$3N + (N-1) + 2(N-2) + \binom{N+d-3}{N} + \binom{N+d-3}{N-3} + \sum_{k=1}^{d-1} \binom{N+d-k-4}{N-3} - 1.$$

26

7.3 Components outside $\mathcal{V}_{d,3}$: Extremal case

We show the following result for the extremal case.

Theorem 7.5. *Let $\mathcal{V}_{ext}$ be the set of configurations $\mathcal{L} \in (\mathcal{P}_d)^3$ that are suited for degree d in $\mathbb{P}^N$ and such that $P_i \in \Pi_j$ for all $i, j \in \{1, 2, 3\}$. If $d \geq 6$, then $\mathcal{V}_{ext}$ has two irreducible components. The first component $\mathcal{V}_{ext,I}$ corresponds to configurations $\mathcal{L}$ with Π_1, Π_2, Π_3 linearly independent and has dimension*

$$3N + 3(N-3) + 3\binom{N+d-4}{N-2} + 3\binom{N+d-5}{N-4} + \binom{N+d-6}{N-6} - 1;$$

the second component $\mathcal{V}_{ext,II}$ corresponds to configurations $\mathcal{L}$ with Π_1, Π_2, Π_3 linearly dependent and has dimension

$$3N + 2(N-3) + \binom{N+d-3}{N-1} + 2\binom{N+d-4}{N-3} + \binom{N+d-5}{N-3} + \binom{N+d-5}{N-5}.$$

Assume $\mathcal{L} \in (\mathcal{P}_d)^3$ with $\mathbb{P}_d(\mathcal{L}) \neq \emptyset$ and $P_i \in \Pi_j$ for all $i, j \in \{1, 2, 3\}$. In this case, it is easy to see that

$$L = \langle P_1, P_2, P_3 \rangle \subset \Pi_1 \cap \Pi_2 \cap \Pi_3 \cap X = C_1 \cap C_2 \cap C_3.$$

Note that $P_3 \notin \langle P_1, P_2 \rangle$ (see Proposition 3.1) and that $N > 4$ (for $N = 4$, the plane L would be a component of the cones C_i, which contradicts Lemma 2.4).

Choose coordinates $(X_0 : \ldots : X_N)$ on $\mathbb{P}^N$ so that $P_1 = (1 : 0 : 0 : \ldots : 0)$, $P_2 = (0 : 1 : 0 \ldots : 0)$, $P_3 = (0 : 0 : 1 : \ldots : 0)$, Π_1 has equation $X_3 = 0$ and Π_2 has equation $X_4 = 0$.

7.3.1 Case I: $\Pi_1 \cap \Pi_2 \not\subset \Pi_3$

We can assume that $\Pi_3 : X_5 = 0$. We can write

$$f = X_3 X_4 X_5 g_{345} + X_3 X_4 g_{34} + X_3 X_5 g_{35} + X_4 X_5 g_{45} + X_3 g_3 + X_4 g_4 + X_5 g_5 + g,$$

where for example g_{34} is independent of X_5, g_3 is independent of X_4 and X_5 and g is independent of X_3, X_4 and X_5.

Since $C_1 \subset \Pi_1$ is defined by $X_4 X_5 g_{45} + X_4 g_4 + X_5 g_5 + g = 0$, we get that g_{45}, g_4, g_5, g are independent of the variable X_0. Analogously, we get that g_{35}, g_3, g_5, g are independent of X_1 and g_{34}, g_3, g_4, g independent of X_2, so we conclude f is of the form

$$X_3 X_4 X_5 g_{345}(X_0, \ldots, X_N) + X_3 X_4 g_{34}(X_0, X_1, X_3, X_4, X_6, \ldots, X_N) +$$
$$X_3 X_5 g_{35}(X_0, X_2, X_3, X_5, \ldots, X_N) + X_4 X_5 g_{45}(X_1, X_2, X_4, X_5, \ldots, X_N) +$$
$$X_3 g_3(X_0, X_3, X_6, \ldots, X_N) + X_4 g_4(X_1, X_4, X_6, \ldots, X_N) +$$
$$X_5 g_5(X_2, X_5, X_6, \ldots, X_N) + g(X_6, \ldots, X_N).$$

When we fix the element $\mathcal{L}$, the polynomials $g_{34}, g_{35}, g_{45}, g_3, g_4, g_5, g$ are fixed, but g_{345} can still vary, so the dimension of $\mathbb{P}_d(\mathcal{L})$ is equal to $\binom{N+d-3}{N}$.

This case gives rise to a component of the set of hypersurfaces with three star points of dimension $d_{\text{ext,I}}$ equal to

$$3N+3(N-3)+\binom{N+d-3}{N}+3\binom{N+d-4}{N-2}+3\binom{N+d-5}{N-4}+\binom{N+d-6}{N-6}-1.$$

7.3.2 Case II: $\Pi_1 \cap \Pi_2 \subset \Pi_3$

We may assume Π_3 is the hyperplane $X_4 - X_3 = 0$. Write f as $X_3 X_4 g_{34} + X_3 g_3 + X_4 g_4 + g$, with g_3 independent of X_4, g_4 of X_3 and g of X_3 and X_4.

Since $C_1 \subset \Pi_1$ is given by $X_4 g_4 + g$, we see that g_4 and g are independent of X_0. By considering $C_2 \subset \Pi_2$, we get that g_3 and g are independent of X_1. The cone C_3 is has equations $X_4 - X_3 = f = 0$, so

$$X_3^2 g_{34}(X_0, X_1, X_2, X_3, \underline{X_3}, X_5, \ldots, X_N) + X_3 g_3(X_0, X_2, X_3, X_5, \ldots, X_N)+$$
$$X_3 g_4(X_1, X_2, \underline{X_3}, X_5, \ldots, X_N) + g(X_2, X_5, \ldots, X_N)$$

is independent of the variable X_2. This implies that g is independent of X_2. If we write $g_i = X_i h_i + h_i'$ for $i \in \{3, 4\}$ with h_i' independent of X_i, we see that

$$h_3'(X_0, X_2, X_5, \ldots, X_N) + h_4'(X_1, X_2, X_5, \ldots, X_N)$$

and

$$g_{34}(X_0, X_1, X_2, X_3, \underline{X_3}, X_5, \ldots, X_N) + h_3(X_0, X_2, X_3, X_5, \ldots, X_N)$$
$$+ h_4(X_1, X_2, \underline{X_3}, X_5, \ldots, X_N)$$

are independent of X_2. If we write $h_3' = X_2 b_2 + b$ and $h_4' = X_2 c_2 + c$ with b and c independent of X_2, we have that $b_2 + c_2 \equiv 0$. Hence, b_2 is independent of X_0 and c_2 is independent of X_1. On the other hand, if $g_{34} = X_2 a_2 + a$, $h_3 = X_2 b_{23} + b_3$ and $h_4 = X_2 c_{24} + c_4$ with a, b_3 and c_4 independent of X_2, we get

$$a_2(X_0, X_1, X_2, X_3, \underline{X_3}, X_5, \ldots, X_N) + b_{23}(X_0, X_2, X_3, X_5, \ldots, X_N)$$
$$+ c_{24}(X_1, X_2, \underline{X_3}, X_5, \ldots, X_N) \equiv 0.$$

We see that $a_2(X_0, X_1, X_2, X_3, \underline{X_3}, X_5, \ldots, X_N)$ does not contain terms divisible by $X_0 X_1$, hence we can write a_2 as $X_0 X_1(X_4 - X_3)a_{012} + X_0 a_{02} + X_1 a_{12} + a'$, with a_{02} independent of X_1, a_{12} independent of X_0 and a_2' independent of X_0 and X_1. Also write b_{23} as $X_0 b_{023} + b_{23}'$ with b_{23}' independent of X_0 and c_{24} as $X_1 c_{124} + c_{24}'$ with c_{24}' independent of X_1. We have that

$$b_{023}(X_0, X_2, X_3, X_5, \ldots, X_N) \equiv -a_{02}(X_0, X_2, X_3, \underline{X_3}, X_5, \ldots, X_N),$$

$$c_{124}(X_1, X_2, \underline{X_3}, X_5, \ldots, X_N) \equiv -a_{12}(X_1, X_2, X_3, \underline{X_3}, X_5, \ldots, X_N)$$

or

$$c_{124}(X_1, X_2, X_4, X_5, \ldots, X_N) \equiv -a_{12}(X_1, X_2, \underline{X_4}, X_4, X_5, \ldots, X_N)$$

28

and

$$b'_{23}(X_2, X_3, X_5, \ldots, X_N) \equiv$$
$$- a'_2(X_2, X_3, \underline{X_3}, X_5, \ldots, X_N) - c'_{24}(X_2, \underline{X_3}, X_5, \ldots, X_N).$$

We conclude that f is of the form

$$X_3 X_4 (X_2 a_2 + a) + X_3 (X_2 X_3 b_{23} + X_3 b_3 + X_2 b_2 + b) +$$
$$X_4 (X_2 X_4 (X_1 c_{124} + c'_{24}) + X_4 c_4 + X_2 c_2 + c) + g,$$

with the above conditions (in particular, b_{23}, b_2 and c_{124} are fixed, given a_2, c_2 and c'_{24}).

In this case, the dimension of $\mathbb{P}_d(\mathcal{L})$ is again $\binom{N+d-3}{N}$. The set of hypersurfaces, corresponding to star point configurations $\mathcal{L} \in (\mathcal{P}_d)^3$ with $P_i \in \Pi_j$ and Π_1, Π_2, Π_3 linearly dependent, gives rise to an irreducible locus of dimension $d_{\mathrm{ext},\mathrm{II}}$ equal to

$$3N + 2(N-3) + \binom{N+d-2}{N} + 2\binom{N+d-4}{N-3} + \binom{N+d-5}{N-3} + \binom{N+d-5}{N-5}.$$

Since

$$d_{\mathrm{ext},\mathrm{II}} = d_{\mathrm{ext},\mathrm{I}} + 4 - N + \binom{N+d-6}{N-1},$$

we have that $d_{\mathrm{ext},\mathrm{II}} > d_{\mathrm{ext},\mathrm{I}}$ for $d \geq 6$. This implies Theorem 7.4. The authors expect that the statement of the theorem also holds for $d \in \{3, 4, 5\}$, although is this case $d_{\mathrm{ext},\mathrm{II}} < d_{\mathrm{ext},\mathrm{I}}$ (except for the case $N = d = 5$, where we have $d_{\mathrm{ext},\mathrm{II}} = d_{\mathrm{ext},\mathrm{I}} = 116$).

Acknowledgements

References

[1] R. Beheshti, *Lines on projective hypersurfaces*, J. Reine. Angew. Math. **592** (2006), 1-21.

[2] I. Cheltsov and J. Park, *Log canonical thresholds and generalized Eckardt points*, Sb. Math. **193** (2002), 149-160.

[3] F. Cools and M. Coppens, *Some general results on plane curves with total inflection points*, Arch. Math. **80** (2007), 73-80.

[4] ———, *Plane curves with 3 or 4 total inflection points*, J. Lond. Math. Soc. **77** (2008), 149-163.

[5] E. Dardanelli and B. Van Geemen, *Hessians and the moduli space of cubic surfaces*, Contemp. Math. **422** (2007), 17-36.

[6] F.E. Eckardt, *Ueber diejenigen Flächen dritter Grades, auf denen sich drei geraden linien in einem Punkte schneiden*, Math. Ann. **10** (1876), 227-272.

[7] J. Harris, R. Mazur, and R. Pandharipande, *Hypersurfaces of low degree*, Duke Math. J. **95** (1998), 125-160.

[8] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977.

[9] J.M. Landsberg and O. Tommasi, *On the Debarre-De Jong and Beheshti-Starr conjectures on hypersurfaces with too many lines*, arXiv: 0810.4158.

[10] T.C. Nguyen, *Star points on cubic surfaces*, Ph-D Thesis, RU Utrecht, 2000.

[11] _____, *Non-singular cubic surfaces with star points*, Vietnam J. Math. **29** (2001), 287-292.

[12] _____, *On boundaries of moduli spaces of non-singular cubic surfaces with star points*, Kodai Math. J. **27** (2004), 57-73.

[13] _____, *On semi-stable, singular cubic surfaces*, Séminaires et Congrès **10** (2005), 373-389.

[14] B. Segre, *The non-singular cubic surfaces*, Oxford University Press, 1942.

[15] D.R. Shreve, *On a certain class of symmetric hypersurfaces*, B. Am. Math. Soc. **45** (1939), 948-951.

[16] A.M. Vermeulen, *Weierstrass points of weight two on curves of genus three*, Ph-D Thesis, Universiteit van Amsterdam, 1983.

[17] H. Yoshihara, *Galois points on quartic surfaces*, J. Math. Soc. Japan **53** (2001), 731-743.

[18] _____, *Galois points for smooth hypersurfaces*, J. Algebra **264** (2003), 520-534.

[19] F. Zak, *Tangents and secants of algebraic varieties*, Transl. Math. Monogr., vol. 127, AMS, 1993.